

21. Some Special Series

Exercise 21.1

1. Question

Find the sum of the following series to n terms:

$$1^3 + 3^3 + 5^3 + 7^3 + \dots$$

Answer

nth term would be = $2n - 1$

$$\text{We know, } 1^3 + 2^3 + 3^3 + 4^3 + 5^3 \dots \dots \dots n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

Therefore,

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 \dots \dots \dots (2n)^3 = \left[\frac{2n(2n+1)}{2} \right]^2 \dots \dots \dots \text{equation 1}$$

$$(1^3 + 3^3 + 5^3 \dots \dots \dots (2n-1)^3) + (2^3 + 4^3 + 6^3 \dots \dots \dots (2n)^3) \\ = \left[\frac{2n(2n+1)}{2} \right]^2$$

$$(1^3 + 3^3 + 5^3 \dots \dots \dots (2n-1)^3) + 2^3(1^3 + 2^3 + 3^3 \dots \dots \dots (2n)^3) = \left[\frac{2n(2n+1)}{2} \right]^2 \dots \dots \dots \text{equation 2}$$

From equation 1

$$2^3(1^3 + 2^3 + 3^3 \dots \dots \dots n^3) = 2^3 \left[\frac{n(n+1)}{2} \right]^2$$

[replace 2n by n]

$$2^3(1^3 + 2^3 + 3^3 \dots \dots \dots n^3) = 2^3 \left[\frac{n(n+1)}{2} \right]^2$$

Substituting in equation 2

$$(1^3 + 3^3 + 5^3 \dots \dots \dots (2n-1)^3) + 2^3 \left[\frac{n(n+1)}{2} \right]^2 = \left[\frac{2n(2n+1)}{2} \right]^2$$

$$(1^3 + 3^3 + 5^3 \dots \dots \dots (2n-1)^3) = \left[\frac{n(2n+1)}{1} \right]^2 - 2^3 \left[\frac{n(n+1)}{2} \right]^2$$

$$(1^3 + 3^3 + 5^3 \dots \dots \dots (2n-1)^3) = \frac{(n)^2(2n+1)^2}{1} - 2^3 \left[\frac{n(n+1)}{2} \right]^2$$

$$(1^3 + 3^3 + 5^3 \dots \dots \dots (2n-1)^3) = \frac{(n)^2}{1} \left[(2n+1)^2 - \frac{2(n+1)^2}{1} \right]$$

$$(1^3 + 3^3 + 5^3 \dots \dots \dots (2n-1)^3) = \frac{(n)^2}{1} [4n^2 + 1 + 4n - 2n^2 - 2 - 4n]$$

$$(1^3 + 3^3 + 5^3 \dots \dots \dots (2n-1)^3) = n^2 [2n^2 - 1]$$

2. Question

Find the sum of the following series to n terms:

$$2^3 + 4^3 + 6^3 + 8^3 + \dots$$

Answer

nth term would be $2n$

$$\text{We know } 1^3 + 2^3 + 3^3 + 4^3 + 5^3 \dots \dots \dots n^3 = \left[\frac{n(n+1)}{2} \right]^2 \dots \dots (1)$$

Therefore,

$$(2^3 + 4^3 + 6^3 \dots \dots \dots (2n)^3) = 2^3(1^3 + 2^3 + 3^3 \dots \dots \dots n^3)$$

Substituting the value from 1

$$(2^3 + 4^3 + 6^3 \dots \dots \dots (2n)^3) = 2^3 \left[\frac{n(n+1)}{2} \right]^2$$

3. Question

Find the sum of the following series to n terms:

$$1.2.5 + 2.3.6 + 3.4.7 + \dots \dots$$

Answer

The n th term be $n(n+1)(n+4)$

Thus we can write $1.2.5 + 2.3.6 + 3.4.7 + \dots \dots$

The general term would be $r(r+1)(r+4)$

$$\sum_{r=1}^n r(r+1)(r+4)$$

$$\sum_{r=1}^n r^3 + 5r^2 + 4r$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots \dots + d_0 \sum 1$$

Thus,

$$\sum_{r=1}^n r^3 + 5r^2 + 4r = \sum_{n=1}^n r^3 + 5 \sum_{n=1}^n r^2 + 4 \sum_{n=1}^n r \dots \dots (1)$$

We know

$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + 3^3 \dots \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots \dots + n = \left[\frac{n(n+1)}{2} \right]$$

Substituting in (1)

$$\begin{aligned} & \sum_{r=1}^n r^3 + 5 \sum_{r=1}^n r^2 + 4 \sum_{r=1}^n r \\ &= \left[\frac{n(n+1)}{2} \right]^2 + 5 \left[\frac{n(n+1)(2n+1)}{6} \right] + 4 \left[\frac{n(n+1)}{2} \right] \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{n^2(n+1)^2}{2^2} \right] + 5 \left[\frac{n(n+1)(2n+1)}{6} \right] + 4 \left[\frac{n(n+1)}{2} \right] \\
&= \frac{3n^2(n+1)^2 + 10n(n+1)(2n+1) + 24n(n+1)}{12} \\
&= \frac{3n^4 + 26n^3 + 57n^2 + 34n}{12}
\end{aligned}$$

4. Question

Find the sum of the following series to n terms:

$$1.2.4 + 2.3.7 + 3.4.10 + \dots$$

Answer

The nth term be $n(n+1)(3n+1)$

$$1.2.4 + 2.3.7 + 3.4.10 + \dots = \sum_{r=1}^n r(r+1)(3r+1)$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} + \dots + d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} + \dots + d_0 \sum 1$$

$$\sum_{r=1}^n r(r+1)(3r+1) = \sum_{r=1}^n 3r^3 + 4r^2 + r$$

$$\sum_{r=1}^n 3r^3 + 4r^2 + r = 3 \sum_{r=1}^n r^3 + 4 \sum_{r=1}^n r^2 + \sum_{r=1}^n r \dots (1)$$

We know

$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n r = 1 + 2 + 3 + \dots + n = \left[\frac{n(n+1)}{2} \right]$$

Thus from (1)

$$\begin{aligned}
&= 3 \left[\frac{n(n+1)}{2} \right]^2 + 4 \left[\frac{n(n+1)(2n+1)}{6} \right] + \left[\frac{n(n+1)}{2} \right] \\
&= \frac{9n^2(n+1)^2 + 8n(n+1)(2n+1) + 6n(n+1)}{12} \\
&= \frac{9n^4 + 34n^3 + 39n^2 + 14n}{12}
\end{aligned}$$

5. Question

Find the sum of the following series to n terms:

$$1 + (1+2) + (1+2+3) + (1+2+3+4) + \dots$$

Answer

The nth term be $\sum_{k=0}^n n - k$

Where $\sum_{k=0}^n n - k = (n-0) + (n-1) + (n-2) + \dots + (n-n)$

$$\sum_{k=0}^n n - k = n \sum_{k=0}^n 1 - \sum_{k=0}^n k$$

$$n \sum_{k=0}^n 1 = 1(k=0\text{th}) + 1(k=1\text{th}) + \dots \dots \dots 1(k=n\text{th}) = n(n+1)$$

Since,

$$\sum_{k=0}^n k = 1 + 2 + 3 + 4 \dots n = \frac{n(n+1)}{2}$$

$$\sum_{k=0}^n n - k = n(n+1) - \frac{n(n+1)}{2} \dots \dots \dots (1)$$

$$1 + (1+2) + (1+2+3) + (1+2+3+4) + \dots \dots \dots \sum_{n=1}^n \sum_{k=0}^n n - k$$

From (1)

$$1 + (1+2) + (1+2+3) + (1+2+3+4) + \dots \dots \dots = \sum_{r=1}^n r(r+1) - \frac{r(r+1)}{2}$$

$$\text{Thus, solving } \sum_{r=1}^n r(r+1) - \frac{r(r+1)}{2}$$

$$\sum_{r=1}^n r(r+1) - \frac{r(r+1)}{2} = \sum_{n=1}^n \frac{r(r+1)}{2}$$

$$\sum_{r=1}^n r(r+1) - \frac{r(r+1)}{2} = \sum_{r=1}^n \frac{(r^2 + r)}{2}$$

$$\text{Solving } \sum_{r=1}^n \frac{(r^2 + r)}{2}$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots \dots \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots \dots \dots + d_0 \sum 1$$

Thus,

$$\sum_{r=1}^n \frac{(r^2 + r)}{2} = \frac{1}{2} \left(\sum_{r=1}^n r^2 + \sum_{n=1}^n r \right)$$

We know,

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots \dots \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots \dots \dots + n = \left[\frac{n(n+1)}{2} \right]$$

Substituting

$$\sum_{r=1}^n \frac{(r^2 + r)}{2} = \frac{1}{2} \left(\left[\frac{n(n+1)(2n+1)}{6} \right] + \left[\frac{n(n+1)}{2} \right] \right)$$

$$= \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} \right] + \left[\frac{n(n+1)}{2} \right]$$

$$= \frac{1n(n+1)}{2(2)} \left[\frac{(2n+1)}{3} + 1 \right]$$

$$= \frac{1n(n+1)}{2(2)} \left[\frac{(2n+4)}{3} \right]$$

$$= \frac{n(n+1)(n+2)}{6}$$

Thus the answer is $\frac{n(n+1)(n+2)}{6}$

6. Question

Find the sum of the following series to n terms:

$$1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \dots$$

Answer

The last term be $n(n+1)$

The generalized equation be

$$\sum_{r=1}^n r(r+1) = \sum_{r=1}^n r^2 + \sum_{r=1}^n r \dots \dots \dots (1)$$

Since We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots \dots + d_0 \sum 1$$

We know

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots \dots + n = \left[\frac{n(n+1)}{2} \right]$$

Thus substituting in (1)

$$= \left[\frac{n(n+1)(2n+1)}{6} \right] + \left[\frac{n(n+1)}{2} \right]$$

$$= \frac{n(n+1)}{2} \left[\frac{(2n+1)}{3} + 1 \right]$$

$$= \frac{n(n+1)}{2} \left[\frac{(2n+4)}{3} \right]$$

$$= \frac{n(n+1)}{1} \left[\frac{(n+2)}{3} \right]$$

7. Question

Find the sum of the following series to n terms:

$$3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \dots$$

Answer

The nth term will be $n^2 \times (2n+1)$

The generalized equation be

$$\sum_{r=1}^n r^2 \times (2r + 1) = \sum_{n=1}^n 2r^3 + r^2$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots + d_0 \sum 1$$

Thus

$$\sum_{r=1}^n 2r^3 + r^2 = 2 \sum_{n=1}^n r^3 + \sum_{n=1}^n r^2 \dots (1)$$

We know

$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + 3^3 \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots + n = \left[\frac{n(n+1)}{2} \right]$$

Substituting the values in (1)

$$\sum_{r=1}^n 2r^3 + r^2 = 2 \left[\frac{n(n+1)}{2} \right]^2 + \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n 2r^3 + r^2 = \frac{n(n+1)}{2} \left[2 \left[\frac{n(n+1)}{2} \right] + \frac{2n+1}{3} \right]$$

$$\sum_{r=1}^n 2r^3 + r^2 = \frac{n(n+1)}{2} \left[\frac{3n(n+1) + 2n+1}{3} \right]$$

$$\sum_{r=1}^n 2r^3 + r^2 = \frac{n(n+1)}{2} \left[\frac{3n^2 + 5n + 1}{3} \right]$$

8 A. Question

Find the sum of the series whose nth term is :

$$2n^3 + 3n^2 - 1$$

Answer

$$1^{\text{st}} \text{ term} = 2(1)^3 + 3(1)^2 - 1$$

$$2^{\text{nd}} \text{ term} = 2(2)^3 + 3(2)^2 - 1$$

And so on

$$N^{\text{th}} \text{ term} = 2n^3 + 3n^2 - 1$$

$$\text{General term be} = 2r^3 + 3r^2 - 1$$

$$\text{Summation} = 1^{\text{st}} \text{ term} + 2^{\text{nd}} \text{ term} + \dots + n^{\text{th}} \text{ term}$$

$$= 2(1)^3 + 3(1)^2 - 1 + 2(2)^3 + 3(2)^2 - 1 + \dots + 2n^3 + 3n^2 - 1 \dots (1)$$

We know,

$$\sum_{x=1}^n f(x) = f(1) + f(2) + \dots + f(n)$$

Thus

From (1) we have

$$\text{Summation} = \sum_{r=1}^n 2r^3 + 3r^2 - 1$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots + d_0 \sum 1$$

Thus

$$\sum_{r=1}^n 2r^3 + 3r^2 - 1 = 2 \sum_{x=1}^n r^3 + 3 \sum_{x=1}^n r^2 - \sum_{x=1}^n 1 \dots (2)$$

We know

$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + 3^3 \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n 1 = 1 + 1 + \dots n \text{ times} = n$$

Thus substituting the above values in (2)

$$\text{Summation} = \sum_{r=1}^n 2n^3 + 3n^2 - 1 = 2 \left[\frac{n(n+1)}{2} \right]^2 + 3 \left[\frac{n(n+1)(2n+1)}{6} \right] - n$$

$$= 2 \left[\frac{n(n+1)}{2} \right]^2 + 3 \left[\frac{n(n+1)(2n+1)}{6} \right] - n$$

$$= \frac{n(n+1)}{2} \left[2 \left[\frac{n(n+1)}{2} \right] + \frac{3[2n+1]}{3} \right] - n$$

$$= \frac{n(n+1)}{2} \left[\frac{n(n+1) + 2n+1}{1} \right] - n$$

$$= \frac{n(n+1)}{2} \left[\frac{n^2 + 3n + 1}{1} \right] - n$$

8 B. Question

Find the sum of the series whose nth term is :

$$n^3 - 3^n$$

Answer

Generalized term be $n^3 - 3^n$

$$1^{\text{st}} \text{ term} = (1)^3 - 3^{(1)}$$

$$2^{\text{nd}} \text{ term} = (2)^3 - 3^{(2)}$$

And so on

$$n^{\text{th}} \text{ term} = n^3 - 3^n$$

$$\text{general term} = r^3 - 3^r$$

$$\text{Summation} = 1^{\text{st}} \text{ term} + 2^{\text{nd}} \text{ term} + \dots + n^{\text{th}} \text{ term}$$

$$= (1)^3 - 3^{(1)} + (2)^3 - 3^{(2)} + \dots + n^3 - 3^n \dots (1)$$

We know

$$\sum_{x=1}^n f(x) = f(1) + f(2) + \dots + f(n)$$

Thus

From (1) we have

$$\text{Summation} = \sum_{r=1}^n r^3 - 3^r$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} + \dots + d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} + \dots + d_0 \sum 1$$

Thus

$$\sum_{r=1}^n r^3 - 3^r = \sum_{r=1}^n r^3 - \sum_{x=1}^n 3^r \dots \dots (2)$$

We know,

$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\sum_{r=1}^n 3^r = 3^1 + 3^2 + 3^3 + \dots + 3^n = \frac{3(3^n - 1)}{3 - 1}$$

$$\text{Since, } \frac{a_1(r^n - 1)}{r - 1} = a_1 + a_2 + a_3 + \dots + a_n \text{ where } r = \frac{a_2}{a_1} \text{ if } \frac{a_2}{a_1} = \frac{a_3}{a_2} = \dots = \frac{a_n}{a_{n-1}}$$

Thus substituting the above values in (2)

$$\sum_{r=1}^n r^3 - 3^r = \left[\frac{n(n+1)}{2} \right]^2 - \frac{3(3^n - 1)}{3 - 1}$$

$$\text{Summation} = \left[\frac{n(n+1)}{2} \right]^2 - \frac{3(3^n - 1)}{3 - 1}$$

$$= \left[\frac{n(n+1)}{2} \right]^2 - \frac{3(3^n - 1)}{2}$$

8 C. Question

Find the sum of the series whose nth term is :

$$n(n+1)(n+4)$$

Answer

Generalized term be $r(r+1)(r+4)$

$$1^{\text{st}} \text{ term} = (1)((1)+1)((1)+4)$$

$$2^{\text{nd}} \text{ term} = (2)((2)+1)((2)+4)$$

And so on

$$\text{nth term} = n(n+1)(n+4) = n^3 + 5n^2 + 4n$$

$$\text{Summation} = 1^{\text{st}} \text{ term} + 2^{\text{nd}} \text{ term} + \dots + \text{nth term}$$

$$= (1)((1)+1)((1)+4) + (2)((2)+1)((2)+4) + \dots + n^3 + 5n^2 + 4n \dots (1)$$

We know,

$$\sum_{x=1}^n f(x) = f(1) + f(2) + \dots + f(n)$$

Thus

From (1) we have

$$\text{Summation} = \sum_{r=1}^n r^3 + 5r^2 + 4r$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} + \dots + d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} + \dots + d_0 \sum 1$$

Thus

$$\sum_{r=1}^n r^3 + 5r^2 + 4r = \sum_{x=1}^n r^3 + 5 \sum_{x=1}^n r^2 + 4 \sum_{x=1}^n r \quad (2)$$

We know,

$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n r = 1 + 2 + 3 + \dots + n = \left[\frac{n(n+1)}{2} \right]$$

Thus substituting the above values in (2)

$$\sum_{r=1}^n r^3 + 5r^2 + 4r = \left[\frac{n(n+1)}{2} \right]^2 + 5 \left[\frac{n(n+1)(2n+1)}{6} \right] + 4 \left[\frac{n(n+1)}{2} \right]$$

$$\text{Summation} = \left[\frac{n(n+1)}{2} \right]^2 + 5 \left[\frac{n(n+1)(2n+1)}{6} \right] + 4 \left[\frac{n(n+1)}{2} \right]$$

$$= \left[\frac{n(n+1)}{2} \right]^2 + 5 \left[\frac{n(n+1)(2n+1)}{6} \right] + 4 \left[\frac{n(n+1)}{2} \right]$$

$$= \frac{n(n+1)}{2} \left\{ \left[\frac{n(n+1)}{2} \right] + 5 \left[\frac{(2n+1)}{3} \right] + 4[1] \right\}$$

$$= \frac{n(n+1)}{2} \left\{ \frac{3n(n+1) + 10(2n+1) + 24}{6} \right\}$$

$$= \frac{n(n+1)}{2} \left\{ \frac{3n^2 + 23n + 34}{6} \right\}$$

8 D. Question

Find the sum of the series whose nth term is :

$$(2n-1)^2$$

Answer

$$\text{Generalized term be } (2r-1)^2 = 4r^2 + 1 - 4r$$

$$1^{\text{st}} \text{ term} = 4(1)^2 + 1 - 4(1)$$

$$2^{\text{nd}} \text{ term} = 4(2)^2 + 1 - 4(2)$$

And so on

$$\text{nth term} = 4n^2 + 1 - 4n$$

$$\text{Summation} = 1^{\text{st}} \text{ term} + 2^{\text{nd}} \text{ term} + \dots + \text{nth term}$$

$$= 4(1)^2 + 1 - 4(1) + 4(2)^2 + 1 - 4(2) \dots\dots\dots 4n^2 + 1 - 4n \dots\dots(1)$$

We know ,

$$\sum_{x=1}^n f(x) = f(1) + f(2) + \dots\dots\dots f(n)$$

Thus

From (1) we have

$$\text{Summation} = \sum_{r=1}^n 4r^2 + 1 - 4r$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots\dots\dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots\dots\dots + d_0 \sum 1$$

Thus

$$\sum_{r=1}^n 1 + 4r^2 - 4r = \sum_{r=1}^n 1 + 4 \sum_{r=1}^n r^2 - 4 \sum_{x=1}^n r \quad (2)$$

We know

$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + 3^3 \dots\dots\dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots\dots\dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots\dots\dots + n = \left[\frac{n(n+1)}{2} \right]$$

$$\sum_{r=1}^n 1 = n$$

Thus substituting above values in (2)

$$\sum_{r=1}^n 1 + 4r^2 - 4r = n + 4 \left[\frac{n(n+1)(2n+1)}{6} \right] - 4 \left[\frac{n(n+1)}{2} \right]$$

$$= n + 4 \left[\frac{n(n+1)(2n+1)}{6} \right] - 4 \left[\frac{n(n+1)}{2} \right]$$

$$= n + \frac{4n(n+1)}{2} \left\{ \left[\frac{(2n+1)}{3} \right] - [1] \right\}$$

$$= n + \frac{4n(n+1)}{2} \left[\frac{2n-2}{6} \right]$$

$$= n + \frac{2n(n+1)}{1} \left[\frac{n-1}{3} \right]$$

9. Question

Find the 20th term and the sum of 20 terms of the series :

$$2 \times 4 + 4 \times 6 + 6 \times 8 + \dots\dots\dots$$

Answer

$$\text{Given: } 2 \times 4 + 4 \times 6 + 6 \times 8 + \dots\dots\dots$$

The nth term would be from given series $2n \times (2n + 2)$

The general term would be from given series $2r \times (2r + 2)$

Thus 20th term be $2(20) \{2(20) + 2\} = 40 \times 42 = 1680$

Summation = 1st term + 2nd term + + 20th term

$$= 2 \times 4 + 4 \times 6 + 6 \times 8 + \dots 40 \times 42 \quad (1)$$

We know

$$\sum_{x=1}^n f(x) = f(1) + f(2) + \dots \dots \dots f(n)$$

Thus

From (1) we have

$$\text{Summation} = \sum_{r=1}^{20} 2r \times (2r + 2)$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots \dots + d_0 \sum 1$$

$$\sum_{r=1}^{20} 2r \times (2r + 2) = \sum_{r=1}^{20} (4r^2 + 4r)$$

Thus,

$$\sum_{r=1}^{20} (4r^2 + 4r) = 4 \sum_{r=1}^{20} r^2 + 4 \sum_{r=1}^{20} r \quad (2)$$

We know

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots \dots + n = \left[\frac{n(n+1)}{2} \right]$$

Thus

$$\sum_{r=1}^{20} r^2 = \left[\frac{20(20+1)(2(20)+1)}{6} \right] = \left[\frac{20(21)(41)}{6} \right]$$

$$\sum_{r=1}^{20} r = \left[\frac{20(20+1)}{2} \right] = \left[\frac{20(21)}{2} \right]$$

Thus substituting in above equation in (2)

$$\sum_{r=1}^{20} (4r^2 + 4r) = 4 \left[\frac{20(21)(41)}{6} \right] + 4 \left[\frac{20(21)}{2} \right]$$

$$= 4(10)(7)(41) + 4(10)(21)$$

$$= 12320$$

1. Question

If the line segment joining the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ subtends an angle α at the origin O, prove that :
 $OP \cdot OQ \cos \alpha = x_1 x_2 + y_1 y_2$.

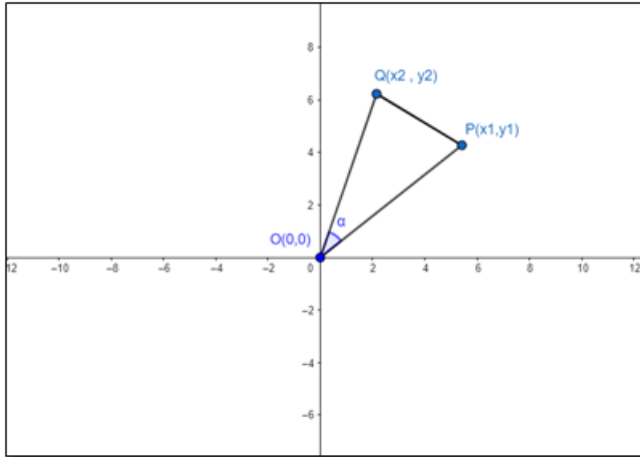
Answer

Key points to solve the problem:

• **The idea of distance formula-** Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by- **PQ** = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Given,

Two points P and Q subtends an angle α at the origin as shown in figure:



From the figure we can see that points O, P and Q forms a triangle.

Clearly in $\triangle OPQ$ we have:

$$\cos \alpha = \frac{OP^2 + OQ^2 - PQ^2}{2OP \cdot OQ} \text{ \{from cosine formula in a triangle\}}$$

$$\Rightarrow 2OP \cdot OQ \cos \alpha = OP^2 + OQ^2 - PQ^2 \text{equation 1}$$

From distance formula we have-

$$OP = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

As, coordinates of O are $(0, 0) \Rightarrow x_2 = 0$ and $y_2 = 0$

Coordinates of P are $(x_1, y_1) \Rightarrow x_1 = x_1$ and $y_1 = y_1$

$$= \sqrt{(x_1 - 0)^2 + (y_1 - 0)^2}$$

$$= \sqrt{x_1^2 + y_1^2}$$

$$\text{Similarly, } OQ = \sqrt{(x_2 - 0)^2 + (y_2 - 0)^2}$$

$$= \sqrt{x_2^2 + y_2^2}$$

$$\text{And, } PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\therefore OP^2 + OQ^2 - PQ^2 = (\sqrt{x_1^2 + y_1^2})^2 + (\sqrt{x_2^2 + y_2^2})^2 - (\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2})^2$$

$$\Rightarrow OP^2 + OQ^2 - PQ^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 - \{(x_2 - x_1)^2 + (y_2 - y_1)^2\}$$

Using $(a-b)^2 = a^2 + b^2 - 2ab$

$$\therefore OP^2 + OQ^2 - PQ^2 = 2x_1 x_2 + 2y_1 y_2 \text{equation 2}$$

From equation 1 and 2 we have:

$$2\vec{OP} \cdot \vec{OQ} \cos \alpha = 2x_1x_2 + 2y_1y_2$$

$$\Rightarrow \vec{OP} \cdot \vec{OQ} \cos \alpha = x_1x_2 + y_1y_2 \dots \text{Proved.}$$

2. Question

The vertices of a triangle ABC are A(0, 0), B (2, -1) and C (9, 0). Find cos B.

Answer

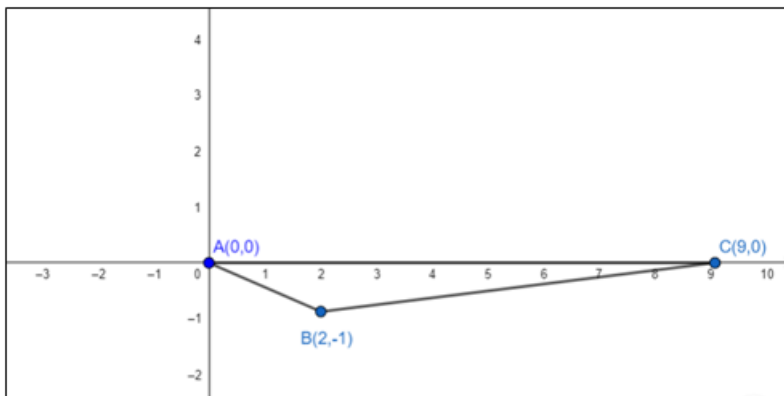
Key points to solve the problem:

• **The idea of distance formula-** Distance between two points P(x₁,y₁) and Q(x₂,y₂) is given by- $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Given,

Coordinates of the triangle and we need to find cos B which can be easily found using cosine formula.

See the figure:



From cosine formula in ΔABC , We have:

$$\cos B = \frac{AB^2 + BC^2 - AC^2}{2AB \cdot BC}$$

using distance formula we have:

$$AB = \sqrt{(2 - 0)^2 + (-1 - 0)^2} = \sqrt{5}$$

$$BC = \sqrt{(9 - 2)^2 + (0 - (-1))^2} = \sqrt{7^2 + 1^2} = \sqrt{50}$$

$$\text{And, } AC = \sqrt{(9 - 0)^2 + (0 - 0)^2} = 9$$

$$\therefore \cos B = \frac{(\sqrt{5})^2 + (\sqrt{50})^2 - 9^2}{2\sqrt{5}\sqrt{50}} = \frac{55 - 81}{2\sqrt{5}\sqrt{2 \times 25}} = \frac{-26}{10\sqrt{10}} = \frac{-13}{5\sqrt{10}} \dots \text{ans}$$

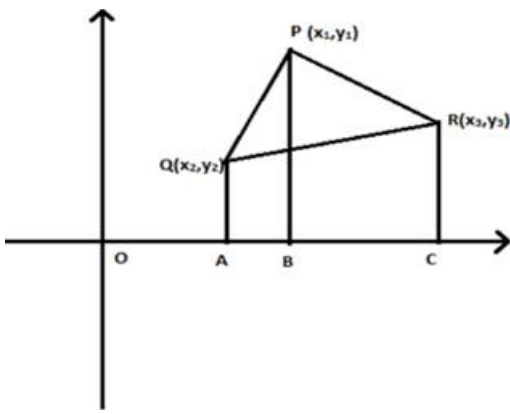
3. Question

Four points A (6, 3), B(-3, 5), C(4, -2) and D(x, 3x) are given in such a way that $\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$, find x.

Answer

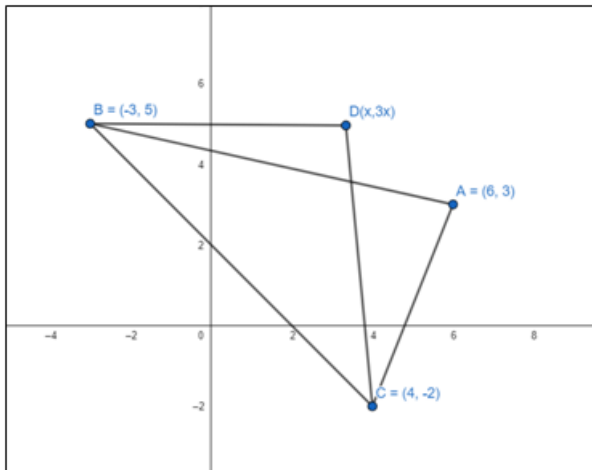
Key points to solve the problem:

• **The idea of distance formula-** Distance between two points P(x₁,y₁) and Q(x₂,y₂) is given by- $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



• **Area of a ΔPQR** - Let $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be the 3 vertices of ΔPQR .

$$Ar(\Delta PQR) = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$



Given, coordinates of the triangle as shown in the figure.

$$\text{Also, } \frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}$$

$$ar(\Delta DBC) = \frac{1}{2}[x(5 - (-2)) + (-3)(-2 - 3x) + 4(3x - 5)]$$

$$= \frac{1}{2}[7x + 6 + 9x + 12x - 20] = 14x - 7$$

$$\text{Similarly, } ar(\Delta ABC) = \frac{1}{2}[6(5 - (-2)) + (-3)(-2 - 3) + 4(3 - 5)]$$

$$= \frac{1}{2}[42 + 15 - 8] = \frac{49}{2} = 24.5$$

$$\therefore \frac{\Delta DBC}{\Delta ABC} = \frac{1}{2} = \frac{14x-7}{24.5}$$

$$\Rightarrow 24.5 = 28x - 14$$

$$\Rightarrow 28x = 38.5$$

$$\Rightarrow x = 38.5/28 = 1.375 \dots \text{ans}$$

4. Question

The points A (2, 0), B(9, 1), C (11, 6) and D (4, 4) are the vertices of a quadrilateral ABCD. Determine whether ABCD is a rhombus or not.

Answer

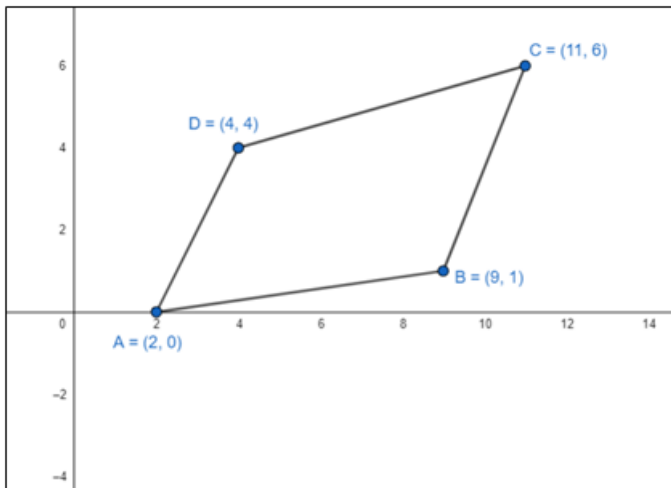
Key points to solve the problem:

• **The idea of distance formula-** Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by- $PQ =$

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

• **The idea of Rhombus** - It is a quadrilateral with all four sides equal.

Given, coordinates of 4 points that form a quadrilateral as shown in fig:



Using distance formula, we have:

$$AB = \sqrt{(9 - 2)^2 + (1 - 0)^2} = \sqrt{7^2 + 1} = \sqrt{50}$$

$$BC = \sqrt{(11 - 9)^2 + (6 - 1)^2} = \sqrt{2^2 + 5^2} = \sqrt{29}$$

Clearly, $AB \neq BC \Rightarrow$ quad ABCD does not have all 4 sides equal.

$\therefore$ ABCD is not a Rhombus ...**ans**

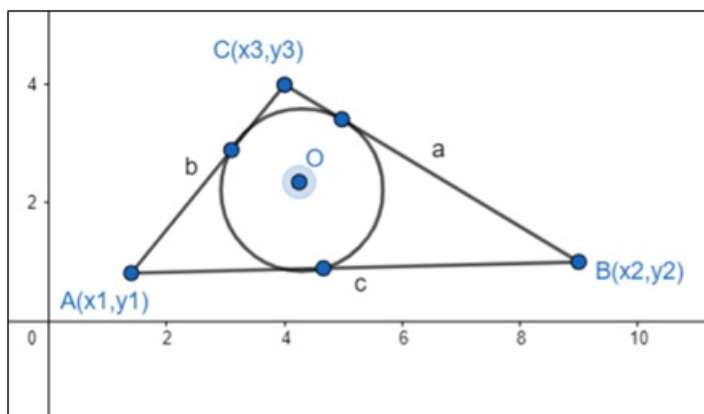
5. Question

Find the coordinates of the centre of the circle inscribed in a triangle whose vertices are (-36, 7), (20, 7) and (0, -8).

Answer

Key points to solve the problem:

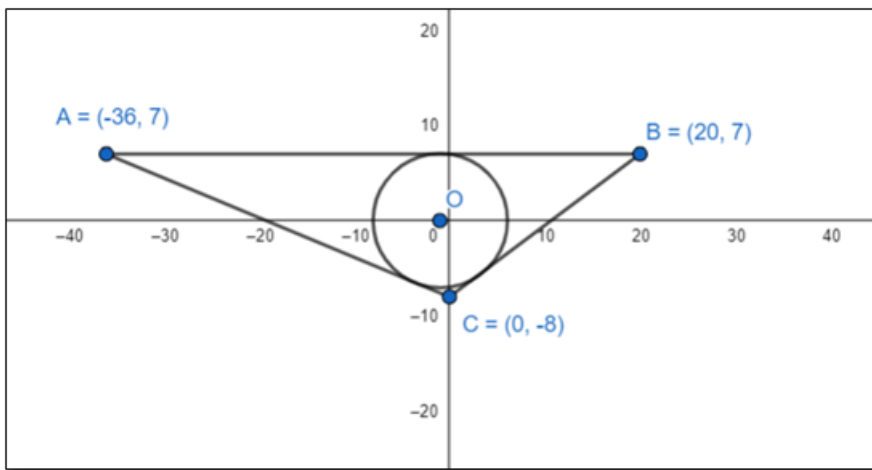
• **The idea of distance formula**- Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by- **PQ** = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



Incentre of a triangle - Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the 3 vertices of $\triangle ABC$ and O be the centre of the circle inscribed in $\triangle ABC$

O = $\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$ where a, b and c are length of sides opposite to $\angle A$, $\angle B$ and $\angle C$ respectively.

Given, coordinates of vertices of the triangle as shown in figure:



We need to find the coordinates of O:

Before that, we have to find a, b and c. We will use the distance formula to find the same.

$$\text{As, } a = BC = \sqrt{(20 - 0)^2 + (7 - (-8))^2} = \sqrt{20^2 + 15^2} = 25$$

$$b = AC = \sqrt{(-36 - 0)^2 + (7 - (-8))^2} = \sqrt{36^2 + 15^2} = \sqrt{1521} = 39$$

$$\text{and } c = AB = \sqrt{(-36 - 20)^2 + (7 - 7)^2} = 56$$

$$\therefore \text{coordinates of O} = \left(\frac{25(-36) + 39(20) + 56(0)}{25 + 39 + 56}, \frac{25(7) + 39(7) + 56(-8)}{25 + 39 + 56} \right)$$

$$= \left(\frac{-1}{120}, \frac{0}{120} \right) = (-1, 0) \dots \text{ans}$$

6. Question

The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find the vertices of the triangle.

Answer

Key points to solve the problem:

• **The idea of distance formula-** Distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by- $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

• **Equilateral triangle-** triangle with all 3 sides equal.

• **Coordinates of the midpoint of a line segment -** Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the end points of line segment PQ. Then coordinated of the midpoint of PQ is given by - $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

Given, an equilateral triangle with base along y axis and midpoint at (0,0)

$\therefore$ coordinates of triangle will be $A(0, y_1)$ $B(0, y_2)$ and $C(x, 0)$

As midpoint is at origin $\Rightarrow y_1 + y_2 = 0 \Rightarrow y_1 = -y_2 \dots \dots \text{eqn 1}$

Also length of each side = $2a$ (given)

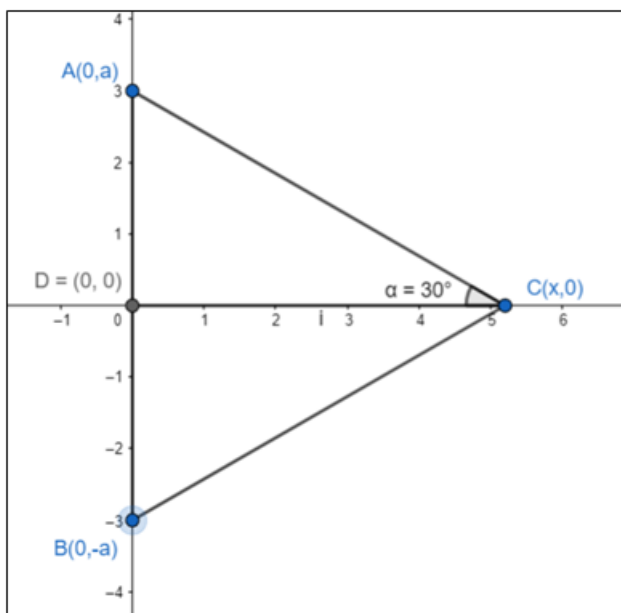
$$\therefore AB = \sqrt{(0 - 0)^2 + (y_2 - y_1)^2} = y_2 - y_1 = 2a \dots \dots \text{eqn 2}$$

$\therefore$ from eqn 1 and 2:

$$y_1 = a \text{ and } y_2 = -a$$

$\therefore$ 2 coordinates are - $A(0, a)$ and $B(0, -a)$

See the figure:



Clearly from figure:

$$DC = x$$

$$\text{Also in } \triangle ADC: \cos 30^\circ = \frac{DC}{AC} = \frac{x}{\sqrt{(0-x)^2 + (a-0)^2}}$$

$$\therefore \frac{\sqrt{3}}{2} = \frac{x}{\sqrt{x^2 + a^2}}$$

Squaring both sides:

$$3(x^2 + a^2) = 4x^2 \Rightarrow x^2 = 3a^2$$

$$\therefore x = \pm\sqrt{3a}$$

$\therefore$ Coordinates of C are $(\sqrt{3}a, 0)$ or $(-\sqrt{3}a, 0)$ ans

7. Question

Find the distance between $P(x_1, y_1)$ and $Q(x_2, y_2)$ when (i) PQ is parallel to the y-axis (ii) PQ is parallel to the x-axis.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by- **AB** = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Given, $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points.

i) When PQ is parallel to the y-axis

This implies that x - coordinate is constant $\Rightarrow x_2 = x_1$

$\therefore$ from distance formula:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{0 + (y_2 - y_1)^2} = |y_2 - y_1| \dots \text{ans}$$

ii) When PQ is parallel to the x-axis

This implies that y - coordinate is constant $\Rightarrow y_2 = y_1$

$\therefore$ from distance formula:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{0 + (x_2 - x_1)^2} = |x_2 - x_1| \dots \text{ans}$$

Note: we take modulus because square root gives both positive and negative values, but distance is always positive so we make it positive using modulus function.

8. Question

Find a point on the x-axis, which is equidistant from the point (7, 6) and (3, 4).

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by- **AB** = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

As the point is on the x-axis so y-coordinate is 0.

Let the coordinate be (x,0)

Given distance of (x,0) from (7,6) and (3,4) is same.

∴ using distance formula we have:

$$\sqrt{(x-7)^2 + (0-6)^2} = \sqrt{(x-3)^2 + (0-4)^2}$$

squaring both sides, we have:

$$(x-7)^2 + (0-6)^2 = (x-3)^2 + (0-4)^2$$

$$\Rightarrow x^2 + 49 - 14x + 36 = x^2 + 9 - 6x + 16$$

$$\Rightarrow 8x = 60 \Rightarrow x = \frac{60}{8} = \frac{15}{2} = 7.5$$

∴ point on x-axis is (7.5,0) ...ans

Exercise 21.2

1. Question

Sum the following series to n terms :

3 + 5 + 9 + 15 + 23 +

Answer

Let $s = 3 + 5 + 9 + 15 + 23 + \dots + n$

By shifting each term by one

$$S = 3 + 5 + 9 + 15 + 23 + \dots + \text{nth} \dots (1)$$

$$S = 3 + 5 + 9 + 15 + \dots + (n-1)\text{th} + \text{nth} \dots (2)$$

by (1) - (2) we get

$$0 = 3 + 2 + 4 + 6 + 8 + \dots + \text{nth} - (n-1)\text{th} - n$$

$$\text{Nth} = 3 + 2 + 4 + 6 + 8 + \dots + 2(n-1)\text{th}$$

$$\text{Nth} = 3 + 2(1 + 2 + 3 + 4 + \dots + (n-1)\text{th}) \dots (3)$$

we know

$$\sum_{r=1}^{n-1} r = 1 + 2 + 3 + \dots + n-1 = \left[\frac{n(n-1)}{2} \right]$$

Substituting the above-given value in (3)

$$nth = 3 + 2\left[\frac{n(n-1)}{2}\right]$$

$$nth = 3 + n^2 - n$$

$$\text{general term} = 3 + r^2 - r$$

thus

$$S = 3 + 5 + 9 + 15 + 23 + \dots + nth = \sum_{r=1}^n 3 + r^2 - r$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots + d_0 \sum 1$$

$$S = \sum_{r=1}^n 3 + r^2 - r = 3 \sum_{r=1}^n 1 + \sum_{r=1}^n r^2 - \sum_{r=1}^n r \quad (4)$$

We know

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots + n = \left[\frac{n(n+1)}{2}\right]$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$\sum_{r=1}^n 1 = 1 + 1 + \dots n \text{ times} = n$$

Thus substituting the above values in(4)

$$S = 3n + \left[\frac{n(n+1)(2n+1)}{6}\right] - \left[\frac{n(n+1)}{2}\right]$$

$$S = 3n + \frac{n(n+1)}{2} \left\{ \left[\frac{(2n+1)}{3}\right] - [1] \right\}$$

$$S = 3n + \frac{n(n+1)}{2} \left\{ \frac{(2n-2)}{3} \right\}$$

$$S = 3n + \frac{n(n+1)}{1} \left\{ \frac{(n-1)}{3} \right\}$$

2. Question

Sum the following series to n terms :

$$2 + 5 + 10 + 17 + 26 + \dots$$

Answer

$$\text{Let } S = 2 + 5 + 10 + 17 + 26 + \dots + n$$

By shifting each term by one

$$S = 2 + 5 + 10 + 17 + 26 + \dots + nth \dots (1)$$

$$S = 2 + 5 + 10 + 17 + \dots + (n-1)th + nth \dots (2)$$

by (1) - (2) we get

$$0 = 2 + 3 + 5 + 7 + 9 + \dots nth - (n-1)th - nth$$

$$Nth = 2 + (3 + 5 + 7 + 9 + \dots 2r + 1) \dots (3)$$

$$Nth = 2 + (\text{summation of first } (n-1)th \text{ term})$$

we know,

$$\sum_{r=1}^{n-1} 2n + 1 = 3 + 5 + 7 \dots \dots \dots + 2n + 1 = [n^2 - 1]$$

Substituting the above given value in (3)

$$nth = n^2 - 1 + 2$$

$$\text{general term} = r^2 - 1 + 2$$

thus

$$S = 2 + 5 + 10 + 17 + 26 + \dots \dots \dots + nth = \sum_{r=1}^n 2 + r^2 - 1$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots \dots \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots \dots \dots + d_0 \sum 1$$

$$S = \sum_{r=1}^n 1 + n^2 = 1 \sum_{r=1}^n 1 + \sum_{r=1}^n r^2 \quad (4)$$

We know

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots \dots \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n 1 = 1 + 1 + \dots \dots \dots n \text{ times} = n$$

Thus substituting the above values in (4)

$$S = n + \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$S = \frac{6n + n(n+1)(2n+1)}{6}$$

3. Question

Sum the following series to n terms :

$$1 + 3 + 7 + 13 + 21 + \dots \dots \dots$$

Answer

$$\text{Let } s = 1 + 3 + 7 + 13 + 21 + \dots \dots \dots + n$$

By shifting each term by one

$$s = 1 + 3 + 7 + 13 + 21 + \dots \dots \dots + nth \quad (1)$$

$$s = 1 + 3 + 7 + 13 + \dots \dots \dots + (n-1)th + nth \quad (2)$$

by (1) - (2) we get

$$0 = 1 + 2 + 4 + 6 + 8 + \dots \dots \dots nth - (n-1)th - nth$$

$$nth = 1 + (2 + 4 + 6 + 8 + \dots \dots \dots 2r) \quad (3)$$

$$nth = 1 + (\text{summation of first } (n-1)th \text{ term})$$

we know

$$\sum_{r=1}^{n-1} 2r = 2 + 4 + 6 + 8 \dots \dots \dots + 2n - 2 = [n(n-1)]$$

Substituting the above given value in (3)

$$nth = 1 + n^2 - n$$

$$\text{general term} = 1 + r^2 - r$$

thus

$$s = 1 + 3 + 7 + 13 + 21 + \dots + nth = \sum_{r=1}^n 1 + r^2 - r$$

We know by property that $\sum ax^n + bx^{n-1} + cx^{n-2} \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots + d_0 \sum 1$

$$s = \sum_{r=1}^n 1 + r^2 - r = \sum_{r=1}^n 1 + \sum_{r=1}^n r^2 - \sum_{r=1}^n r \quad (4)$$

We know

$$\sum_{r=1}^n r = 1 + 2 + 3 + \dots + n = \left[\frac{n(n+1)}{2} \right]$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n 1 = 1 + 1 + \dots + n \text{ times} = n$$

Thus substituting the above values in (4)

$$s = n + \left[\frac{n(n+1)(2n+1)}{6} \right] - \left[\frac{n(n+1)}{2} \right]$$

$$s = n + \frac{n(n+1)}{2} \left\{ \left[\frac{(2n+1)}{3} \right] - [1] \right\}$$

$$s = n + \frac{n(n+1)}{2} \left\{ \frac{(2n-2)}{3} \right\}$$

$$s = n + \frac{n(n+1)}{1} \left\{ \frac{(n-1)}{3} \right\}$$

4. Question

Sum the following series to n terms :

$$3 + 7 + 14 + 24 + 37 + \dots$$

Answer

$$\text{Let } S = 3 + 7 + 14 + 24 + 37 + \dots$$

By Shifting each term by one, we get,

$$S = 3 + 7 + 14 + 24 + 37 + \dots + \text{nth term} \dots (1)$$

$$S = 3 + 7 + 14 + 24 + \dots + (n-1)\text{th term} + \text{nth term} \dots (2)$$

Subtracting equation 2 from equation 1 we get,

$$0 = 3 + 4 + 7 + 10 + 13 + \dots + (\text{nth term} - (n-1)\text{th term}) - \text{nth term}$$

$$N\text{th term} = 3 + 4 + 7 + 10 + \dots + \text{nth term} - (n-1)\text{th term}$$

We can see that 3, 4, 7, ... is an A.P with first term = 3 and common difference = 3

$$\text{Sum of this A.P} = \frac{n}{2} [2 \times 3 + (n-1)3] = \frac{n}{2} (3n+3)$$

Therefore,

$$S = \frac{n}{2}(3n + 3) - (n - 1)\text{th term}$$

$$(n - 1)\text{th term} = a + (n - 2)d$$

$$(n - 1)\text{th term} = 3 + (n - 2)3$$

$$(n - 1)\text{th term} = 3n - 3$$

Therefore,

$$S = \frac{n}{2}(3n + 3) - (3n - 3)$$

$$S = \frac{3n^2 - 3n + 6}{2}$$

5. Question

Sum the following series to n terms :

$$1 + 3 + 6 + 10 + 15 + \dots\dots\dots$$

Answer

$$\text{Let } s = 1 + 3 + 6 + 10 + 15 + \dots\dots\dots + n$$

By shifting each term by one

$$S = 1 + 3 + 6 + 10 + 15 + \dots\dots\dots + \text{nth} \dots\dots\dots (1)$$

$$S = 1 + 3 + 6 + 10 + \dots\dots\dots + (n - 1)\text{th} + \text{nth} \dots\dots(2)$$

by (1) - (2) we get

$$0 = 1 + (2 + 3 + 4 + 5 + \dots\dots\text{nth} - (n - 1)\text{th} - n)$$

$$\text{Nth} = 1 + (2 + 3 + 4 + 5 + \dots\dots\text{nth} - (n - 1)\text{th} - \text{nth})$$

$$\text{Nth} = 1 + (2 + 3 + 4 + \dots\dots r + 1) \dots\dots\dots(3)$$

$$\text{Nth} = 1 + (\text{summation upto } (n - 1)\text{th term})$$

we know

$$\sum_{r=1}^{n-1} r + 1 = 2 + 3 \dots\dots\dots + n = \left[\frac{n(n-1)}{2} + n - 1 \right]$$

Substituting the above-given value in (3)

$$\text{nth} = 1 + \left[\frac{n(n-1)}{2} + n - 1 \right]$$

$$\text{nth} = \left[\frac{n(n-1)}{2} + n \right]$$

$$\text{nth} = \left[\frac{n(n+1)}{2} \right]$$

thus

$$S = 3 + 5 + 9 + 15 + 23 + \dots\dots\dots + \text{nth} = \sum_{r=1}^n \left[\frac{r(r+1)}{2} \right]$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots\dots\dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots\dots\dots + d_0 \sum 1$$

$$s = \frac{1}{2} \sum_{r=1}^n r^2 + r = \frac{1}{2} \sum_{r=1}^n r^2 + \frac{1}{2} \sum_{r=1}^n 2r \dots\dots (4)$$

We know

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots \dots \dots + n = \left[\frac{n(n+1)}{2} \right]$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots \dots \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n 1 = 1 + 1 + \dots \dots \dots n \text{ times} = n$$

Thus substituting the above values in(4)

$$s = \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} \right] + \frac{1}{2} \left[\frac{n(n+1)}{2} \right]$$

$$s = \frac{n(n+1)}{4} \left\{ \left[\frac{(2n+1)}{3} \right] + [1] \right\}$$

$$s = \frac{n(n+1)}{4} \left\{ \frac{(2n+4)}{3} \right\}$$

$$s = \frac{n(n+1)}{2} \left\{ \frac{(n+2)}{3} \right\}$$

6. Question

Sum the following series to n terms :

$$1 + 4 + 13 + 40 + 121 + \dots \dots \dots$$

Answer

$$\text{Let } s = 1 + 4 + 13 + 40 + 121 + \dots \dots \dots + n$$

By shifting each term by one

$$S = 1 + 4 + 13 + 40 + 121 + \dots \dots \dots + n^{\text{th}} \dots (1)$$

$$S = 1 + 4 + 13 + 40 + \dots \dots \dots + (n-1)^{\text{th}} + n^{\text{th}} \dots (2)$$

by (1) - (2) we get

$$0 = 1 + (3 + 9 + 27 + 81 + \dots \dots n^{\text{th}} - (n-1)^{\text{th}} - n)$$

$$N^{\text{th}} = 1 + (3 + 3^2 + 3^3 + 3^4 + \dots \dots n^{\text{th}} - (n-1)^{\text{th}} - n^{\text{th}})$$

$$N^{\text{th}} = 1 + (3 + 3^2 + 3^3 + \dots \dots 3^{n-1}) \dots \dots (3)$$

we know

$$\sum_{r=0}^{n-1} 3^r = 1 + 3 + 3^2 \dots \dots \dots + 3^{n-1} = \left[\frac{1(3^n - 1)}{3 - 1} \right]$$

Substituting the above-given value in (3)

$$n^{\text{th}} = \left[\frac{1(3^n - 1)}{3 - 1} \right]$$

$$n^{\text{th}} = \left[\frac{(3^n - 1)}{2} \right]$$

thus

$$s = 1 + 4 + 13 + 40 + 121 + \dots \dots \dots + \left[\frac{(3^n - 1)}{2} \right]$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots + d_0 \sum 1$$

$$s = \frac{1}{2} \sum_{r=1}^n 3^r - 1 = \frac{1}{2} \sum_{r=1}^n 3^r - \frac{1}{2} \sum_{r=1}^n 1 \dots \dots \dots (4)$$

We know

$$\sum_{r=1}^n 3^r = 3 + 3^2 \dots 3^n = (3^n - 1) \frac{3}{3-1}$$

$$\sum_{r=1}^n 3^r = 3 + 3^2 \dots 3^n = (3^n - 1) \frac{3}{2}$$

$$\sum_{r=1}^n 1 = 1 + 1 + \dots \dots n \text{ times} = n$$

Thus substituting the above values in(4)

$$s = \frac{1}{2} \left[(3^n - 1) \frac{3}{3-1} \right] - \frac{1}{2} [n]$$

$$s = \frac{1}{2} \left\{ \left[(3^n - 1) \frac{3}{2} \right] - [n] \right\}$$

7. Question

Sum the following series to n terms :

$$4 + 6 + 9 + 13 + 18 + \dots \dots \dots$$

Answer

$$\text{Let } s = 4 + 6 + 9 + 13 + 18 + \dots \dots \dots + n$$

shifting each term by one,

$$s = 4 + 6 + 9 + 13 + 18 + \dots \dots \dots + n^{\text{th}} \dots (1)$$

$$s = 4 + 6 + 9 + 13 + 18 + \dots \dots \dots + (n-1)^{\text{th}} + n^{\text{th}} \dots (2)$$

by (1) - (2) we get

$$0 = 4 + (2 + 3 + 4 + 5 + \dots \dots n^{\text{th}} - (n-1)^{\text{th}} - n^{\text{th}})$$

$$N^{\text{th}} = 4 + (2 + 3 + 4 + 5 + \dots \dots n^{\text{th}} - (n-1)^{\text{th}})$$

$$N^{\text{th}} = 4 + (2 + 3 + 4 + \dots \dots r + 1) \dots \dots (3)$$

$$N^{\text{th}} = 4 + (\text{summation upto } (n-1)^{\text{th}} \text{ term})$$

we know

$$\sum_{r=1}^{n-1} r + 1 = 2 + 3 \dots \dots + n = \left[\frac{n(n-1)}{2} + n - 1 \right]$$

Substituting the above-given value in (3)

$$n^{\text{th}} = 4 + \left[\frac{n(n-1)}{2} + n - 1 \right]$$

$$n^{\text{th}} = \left[\frac{(n+2)(n-1)}{2} + 4 \right]$$

thus

$$s = 4 + 6 + 9 + 13 + 18 + \dots + \text{nth} = \sum_{r=1}^n \left[\frac{(r+2)(r-1)}{2} + 4 \right]$$

$$s = 4 + 6 + 9 + 13 + 18 + \dots + \text{nth} = \sum_{r=1}^n \left[\frac{r^2 + r}{2} + 3 \right]$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots + d_0 \sum 1$$

$$s = \sum_{r=1}^n \frac{r^2 + r}{2} + 3 = \frac{1}{2} \sum_{r=1}^n r^2 + \frac{1}{2} \sum_{r=1}^n r + 3 \sum_{r=1}^n 1 \quad (4)$$

We know

$$\sum_{r=1}^n r = 1 + 2 + 3 + \dots + n = \left[\frac{n(n+1)}{2} \right]$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n 1 = 1 + 1 + \dots + n \text{ times} = n$$

Thus substituting the above values in(4)

$$s = \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} \right] + \frac{1}{2} \left[\frac{n(n+1)}{2} \right] + 3n$$

$$s = \frac{n(n+1)}{4} \left\{ \left[\frac{(2n+1)}{3} + 1 \right] \right\} + 3n$$

$$s = \frac{n(n+1)}{2} \left\{ \frac{(n+2)}{3} \right\} + 3n$$

$$s = \frac{n(n+1)}{2} \left\{ \frac{(n+2)}{3} \right\} + 3n$$

8. Question

Sum the following series to n terms :

$$2 + 4 + 7 + 11 + 16 + \dots$$

Answer

$$\text{Let } S = 2 + 4 + 7 + 11 + 16 + \dots + n$$

By shifting each term by one

$$S = 2 + 4 + 7 + 11 + 16 + \dots + \text{nth} \dots (1)$$

$$S = 2 + 4 + 7 + 11 + 16 + \dots + (n-1)\text{th} + \text{nth} \dots (2)$$

by (1) - (2) we get

$$0 = 2 + (2 + 3 + 4 + 5 + \dots + \text{nth} - (n-1)\text{th} - \text{nth})$$

$$N\text{th} = 2 + (2 + 3 + 4 + 5 + \dots + \text{nth} - (n-1)\text{th})$$

$$n\text{th} = 2 + (2 + 3 + 4 + \dots + r + 1) \dots (3)$$

$$n\text{th} = 2 + (\text{summation upto } (n-1)\text{th term})$$

we know

$$\sum_{r=1}^{n-1} r + 1 = 2 + 3 + \dots + n = \left[\frac{n(n-1)}{2} + n - 1 \right]$$

Substituting the above-given value in (3)

$$nth = 2 + \left[\frac{n(n-1)}{2} + n - 1 \right]$$

$$nth = \left[\frac{(n+2)(n-1)}{2} + 2 \right]$$

thus

$$s = 2 + 4 + 7 + 11 + 16 + \dots + nth = \sum_{r=1}^n \left[\frac{(r+2)(r-1)}{2} + 2 \right]$$

$$s = 2 + 4 + 7 + 11 + \dots + nth = \sum_{r=1}^n \left[\frac{r^2 + r}{2} + 1 \right]$$

We know by property that:

$$\sum a x^n + b x^{n-1} + c x^{n-2} + \dots + d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} + \dots + d_0 \sum 1$$

$$s = \sum_{r=1}^n \frac{r^2 + r}{2} + 1 = \frac{1}{2} \sum_{r=1}^n r^2 + \frac{1}{2} \sum_{r=1}^n r + \sum_{r=1}^n 1 \dots (4)$$

We know

$$\sum_{r=1}^n r = 1 + 2 + 3 + \dots + n = \left[\frac{n(n+1)}{2} \right]$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n 1 = 1 + 1 + \dots + n \text{ times} = n$$

Thus substituting the above values in(4)

$$s = \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} \right] + \frac{1}{2} \left[\frac{n(n+1)}{2} \right] + n$$

$$s = \frac{n(n+1)}{4} \left\{ \left[\frac{(2n+1)}{3} + 1 \right] \right\} + n$$

$$s = \frac{n(n+1)}{2} \left\{ \frac{(n+2)}{3} \right\} + n$$

$$s = \frac{n(n+1)}{2} \left\{ \frac{(n+2)}{3} \right\} + n$$

9. Question

Sum the following series to n terms :

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots$$

Answer

The general term would be $\frac{1}{(3r-2)(3r+1)}$

The nth term would be $\frac{1}{(3n-2)(3n+1)}$

$$\begin{aligned} \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} \\ = \left[\frac{4-1}{3.1.4} + \frac{7-4}{3.4.7} + \frac{10-7}{3.7.10} + \dots + \frac{(3n+1)-(3n-2)}{3.(3n-2)(3n+1)} \right] \end{aligned}$$

$$\begin{aligned} \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} \\ = \frac{1}{3} \left[\frac{4-1}{1.4} + \frac{7-4}{4.7} + \frac{10-7}{7.10} + \dots + \frac{(3n+1)-(3n-2)}{(3n-2)(3n+1)} \right] \end{aligned}$$

$$\begin{aligned} \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} \\ = \frac{1}{3} \left[1 + \frac{-1}{4} + \frac{1}{4} + \frac{-1}{7} + \frac{1}{7} + \frac{1}{10} + \dots + \frac{1}{3n-2} - \frac{1}{(3n+1)} \right] \end{aligned}$$

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{1}{3} \left[1 - \frac{1}{(3n+1)} \right] = \frac{n}{3n+1}$$

10. Question

Sum the following series to n terms :

$$\frac{1}{1.6} + \frac{1}{6.11} + \frac{1}{11.16} + \frac{1}{16.21} + \dots + \frac{1}{(5n-4)(5n+1)}$$

Answer

The general term would be $\frac{1}{(5r-4)(5r+1)}$

The nth term would be $\frac{1}{(5n-4)(5n+1)}$

$$\begin{aligned} \frac{1}{1.6} + \frac{1}{6.11} + \frac{1}{11.16} + \dots + \frac{1}{(5n-4)(5n+1)} \\ = \left[\frac{6-1}{5.1.6} + \frac{11-6}{5.6.11} + \frac{16-11}{5.11.16} + \dots + \frac{(5n+1)-(5n-4)}{5.(5n-4)(5n+1)} \right] \end{aligned}$$

$$\begin{aligned} \frac{1}{1.6} + \frac{1}{6.11} + \frac{1}{11.16} + \dots + \frac{1}{(5n-4)(5n+1)} \\ = \frac{1}{5} \left[\frac{6-1}{1.6} + \frac{11-6}{6.11} + \frac{16-11}{11.16} + \dots + \frac{(5n+1)-(5n-4)}{(5n-4)(5n+1)} \right] \end{aligned}$$

$$\begin{aligned} \frac{1}{1.6} + \frac{1}{6.11} + \frac{1}{11.16} + \dots + \frac{1}{(5n-4)(5n+1)} \\ = \frac{1}{5} \left[1 + \frac{-1}{6} + \frac{1}{6} + \frac{-1}{11} + \frac{1}{11} + \frac{-1}{16} + \frac{1}{16} + \dots + \frac{1}{5n-4} - \frac{1}{(5n+1)} \right] \end{aligned}$$

$$\frac{1}{1.6} + \frac{1}{6.11} + \frac{1}{11.16} + \dots + \frac{1}{(5n-4)(5n+1)} = \frac{1}{5} \left[1 - \frac{1}{(5n+1)} \right] = \frac{n}{5n+1}$$

1. Question

Find the locus of a point equidistant from the point (2, 4) and the y-axis.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x₁,y₁) and B(x₂,y₂) is given by- **AB =**

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k)

As we need to maintain the same distance of (h,k) from (2,4) and y-axis.

So we select a point (0,k) on the y-axis.

From distance formula:

$$\text{Distance of (h,k) from (2,4)} = \sqrt{(h-2)^2 + (k-4)^2}$$

$$\text{Distance of (h,k) from (0,k)} = \sqrt{(h-0)^2 + (k-k)^2}$$

According to question both distances are same.

$$\therefore \sqrt{(h-2)^2 + (k-4)^2} = \sqrt{(h-0)^2 + (k-k)^2}$$

Squaring both sides:

$$(h-2)^2 + (k-4)^2 = (h-0)^2 + (k-k)^2$$

$$\Rightarrow h^2 + 4 - 4h + k^2 - 8k + 16 = h^2 + 0$$

$$\Rightarrow k^2 - 4h - 8k + 20 = 0$$

Replace (h,k) with (x,y)

Thus, the locus of point equidistant from (2,4) and the y-axis is-

$$y^2 - 4x - 8y + 20 = 0 \text{ans}$$

2. Question

Find the equation of the locus of a point which moves such that the ratio of its distance from (2, 0) and (1, 3) is 5 : 4.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x₁,y₁) and B(x₂,y₂) is given by- **AB =**

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the point whose locus is to be determined to be (h,k)

$$\text{Distance of (h,k) from (2,0)} = \sqrt{(h-2)^2 + (k-0)^2}$$

$$\text{Distance of (h,k) from (1,3)} = \sqrt{(h-1)^2 + (k-3)^2}$$

According to the question:

$$\frac{\sqrt{(h-2)^2 + (k-0)^2}}{\sqrt{(h-1)^2 + (k-3)^2}} = \frac{5}{4}$$

Squaring both sides:

$$16\{(h-2)^2 + k^2\} = 25\{(h-1)^2 + (k-3)^2\}$$

$$\Rightarrow 16\{h^2 + 4 - 4h + k^2\} = 25\{h^2 - 2h + 1 + k^2 - 6k + 9\}$$

$$\Rightarrow 9h^2 + 9k^2 + 14h - 150k + 186 = 0$$

Replace (h,k) with (x,y)

Thus, the locus of a point which moves such that the ratio of its distance from (2, 0) and (1, 3) is 5 : 4 is –

$$9x^2 + 9y^2 + 14x - 150y + 186 = 0 \text{ans}$$

3. Question

A point moves as so that the difference of its distances from (ae, 0) and (-ae, 0) is 2a, prove that the

equation to its locus is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $b^2 = a^2(e^2 - 1)$.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by- **AB** = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

How to approach: To find locus of a point we first assume the coordinate of point to be (h, k) and write a mathematical equation as per the conditions mentioned in question and finally replace (h, k) with (x, y) to get the locus of point.

Let the point whose locus is to be determined be (h,k)

$$\text{Distance of (h,k) from (ae,0)} = \sqrt{(h - ae)^2 + (k - 0)^2}$$

$$\text{Distance of (h,k) from (-ae,0)} = \sqrt{(h - (-ae))^2 + (k - 0)^2}$$

According to question:

$$\sqrt{(h - ae)^2 + (k - 0)^2} - \sqrt{(h - (-ae))^2 + (k - 0)^2} = 2a$$

$$\Rightarrow \sqrt{(h - ae)^2 + (k - 0)^2} = 2a + \sqrt{(h + ae)^2 + (k - 0)^2}$$

Squaring both sides:

$$(h - ae)^2 + (k - 0)^2 = \left\{ 2a + \sqrt{(h + ae)^2 + (k - 0)^2} \right\}^2$$

$$\Rightarrow h^2 + a^2e^2 - 2aeh + k^2 = 4a^2 + \{(h + ae)^2 + k^2\} + 4a\sqrt{(h + ae)^2 + (k - 0)^2}$$

$$\Rightarrow h^2 + a^2e^2 - 2aeh + k^2 = 4a^2 + h^2 + 2aeh + a^2e^2 + k^2 + 4a\sqrt{(h + ae)^2 + (k - 0)^2}$$

$$\Rightarrow -4aeh - 4a^2 = 4a\sqrt{(h + ae)^2 + (k - 0)^2}$$

$$\Rightarrow -4a(eh + a) = 4a\sqrt{(h + ae)^2 + (k - 0)^2}$$

Again squaring both sides:

$$(eh + a)^2 = (h + ae)^2 + (k - 0)^2$$

$$\Rightarrow e^2h^2 + a^2 + 2aeh = h^2 + a^2e^2 + 2aeh + k^2$$

$$\Rightarrow h^2(e^2 - 1) - k^2 = a^2(e^2 - 1)$$

$$\therefore \frac{h^2}{a^2} - \frac{k^2}{a^2(e^2 - 1)} = 1$$

$$\Rightarrow \frac{h^2}{a^2} - \frac{k^2}{b^2} = 1 \text{ where } b^2 = a^2(e^2 - 1)$$

Replace (h,k) with (x,y)

Thus, the locus of a point such that the difference of its distances from (ae, 0) and (-ae, 0) is 2a:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ where } b^2 = a^2(e^2 - 1) \text{proved}$$

4. Question

Find the locus of a point such that the sum of its distances from (0, 2) and (0, -2) is 6.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by- **AB =**
 $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the point whose locus is to be determined to be (h,k)

$$\text{Distance of (h,k) from (0,2)} = \sqrt{(h-0)^2 + (k-2)^2}$$

$$\text{Distance of (h,k) from (0,-2)} = \sqrt{(h-0)^2 + (k-(-2))^2}$$

According to the question:

$$\sqrt{(h)^2 + (k-2)^2} + \sqrt{(h)^2 + (k+2)^2} = 6$$

$$\Rightarrow \sqrt{(h)^2 + (k-2)^2} = 6 - \sqrt{(h)^2 + (k+2)^2}$$

Squaring both sides:

$$h^2 + (k-2)^2 = \{6 - \sqrt{h^2 + (k+2)^2}\}^2$$

$$\Rightarrow h^2 + 4 - 4k + k^2 = 36 + \{h^2 + k^2 + 4k + 4\} - 12\sqrt{h^2 + (k+2)^2}$$

$$\Rightarrow -8k - 36 = -12\sqrt{h^2 + (k+2)^2}$$

$$\Rightarrow -4(2k + 9) = -12\sqrt{h^2 + (k+2)^2}$$

Again squaring both sides:

$$(2k + 9)^2 = \{3\sqrt{h^2 + (k+2)^2}\}^2$$

$$\Rightarrow 4k^2 + 81 + 36k = 9(h^2 + k^2 + 4k + 4)$$

$$\Rightarrow 9h^2 + 5k^2 = 45$$

Replace (h,k) with (x,y)

Thus, the locus of a point such that sum of its distances from (0,2) and (0,-2) is 6:

$$9x^2 + 5y^2 = 45 \text{proved}$$

5. Question

Find the locus of a point which is equidistant from (1, 3) and x-axis.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by- **$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$**

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k)

As we need to maintain the same distance of (h, k) from $(2, 4)$ and x -axis.

So we select a point $(h, 0)$ on the x -axis.

From distance formula:

$$\text{Distance of } (h, k) \text{ from } (1, 3) = \sqrt{(h - 1)^2 + (k - 3)^2}$$

$$\text{Distance of } (h, k) \text{ from } (h, 0) = \sqrt{(h - h)^2 + (k - 0)^2}$$

According to question both distances are same.

$$\therefore \sqrt{(h - 1)^2 + (k - 3)^2} = \sqrt{(h - h)^2 + (k - 0)^2}$$

Squaring both sides:

$$(h - 1)^2 + (k - 3)^2 = (h - h)^2 + (k - 0)^2$$

$$\Rightarrow h^2 + 1 - 2h + k^2 - 6k + 9 = k^2 + 0$$

$$\Rightarrow h^2 - 2h - 6k + 10 = 0$$

Replace (h, k) with (x, y)

Thus, the locus of a point equidistant from $(1, 3)$ and x -axis is-

$$x^2 - 2x - 6y + 10 = 0 \text{ans}$$

6. Question

Find the locus of a point which moves such that its distance from the origin is three times is the distance from the x -axis.

Answer**Key points to solve the problem:**

• **Idea of distance formula-** Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by- **$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$**

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k)

As we need to maintain a distance of (h, k) from origin such that it is 3 times the distance from the x -axis.

So we select a point $(h, 0)$ on the x -axis.

From distance formula:

$$\text{Distance of } (h, k) \text{ from } (0, 0) = \sqrt{(h - 0)^2 + (k - 0)^2}$$

$$\text{Distance of } (h, k) \text{ from } (h, 0) = \sqrt{(h - h)^2 + (k - 0)^2}$$

According to question both distances are same.

$$\therefore \sqrt{(h-0)^2 + (k-0)^2} = 3\sqrt{(h-h)^2 + (k-0)^2}$$

Squaring both sides:

$$h^2 + k^2 = 9k^2$$

$$\Rightarrow h^2 = 8k^2$$

Replace (h,k) with (x,y)

Thus, the locus of a point is $x^2 = 8y^2$ ans

7. Question

A(5, 3), B(3, -2) are two fixed points, find the equation to the locus of a point P which moves so that the area of the triangle PAB is 9 units.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points A(x₁,y₁) and B(x₂,y₂) is given by- **AB =**
 $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

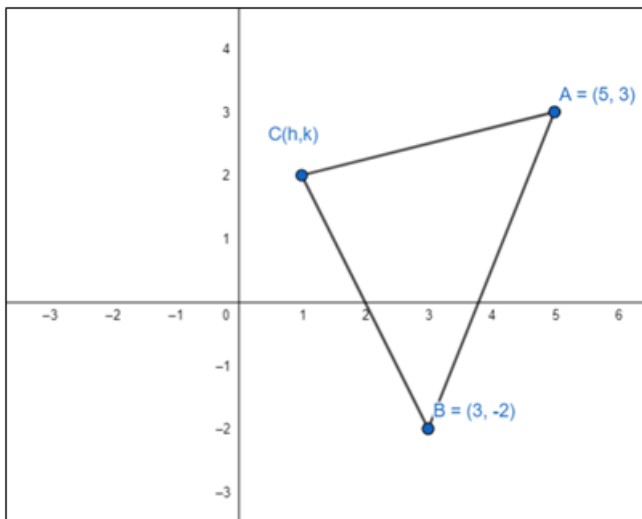
Area of a ΔPQR - Let P(x₁,y₁) , Q(x₂,y₂) and R(x₃,y₃) be the 3 vertices of ΔPQR.

$$Ar(\Delta PQR) = \frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k). Name the moving point to be C

Given the area of ΔABC = 9



According to question:

$$9 = \frac{1}{2} |5(-2 - k) + 3(k - 3) + h((3 - (-2)))|$$

$$\Rightarrow 18 = |-10 - 5k + 3k - 9 + 3h + 2h|$$

$$\Rightarrow |5h - 2k - 19| = 18$$

$$\therefore 5h - 2k - 19 = 18 \text{ or } 5h - 2k - 19 = -18$$

$$\Rightarrow 5h - 2k - 37 = 0 \text{ or } 5h - 2k - 1 = 0$$

Replace (h,k) with (x,y)

Thus, locus of point is $5x-2y-37=0$ or $5x-2y-1=0$ ans

8. Question

Find the locus of a point such that the line segments having end points (2, 0) and (-2, 0) subtend a right angle at that point.

Answer

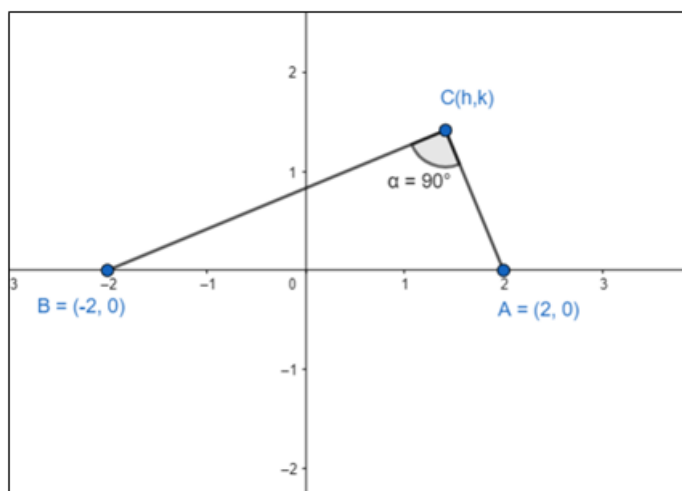
Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by- $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

• **Pythagoras theorem:** In right triangle ΔABC : the sum of the square of two sides is equal to the square of its hypotenuse.

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k) and name the moving point to be C.



According to a question on drawing the figure, we get a right triangle ΔABC .

From Pythagoras theorem we have:

$$BC^2 + AC^2 = AB^2$$

From distance formula:

$$BC = \sqrt{(h - (-2))^2 + (k - 0)^2}$$

$$AC = \sqrt{(h - 2)^2 + (k - 0)^2}$$

$$\text{And } AB = 4$$

$$\therefore \left\{ \sqrt{(h - (-2))^2 + (k - 0)^2} \right\}^2 + \left\{ \sqrt{(h - 2)^2 + (k - 0)^2} \right\}^2 = 16$$

$$\Rightarrow (h + 2)^2 + k^2 + (h - 2)^2 + k^2 = 16$$

$$\Rightarrow h^2 + 4 + 4h + k^2 + h^2 - 4h + 4 + k^2 = 16$$

$$\Rightarrow 2h^2 + 2k^2 - 8 = 0$$

$$\Rightarrow h^2 + k^2 = 4$$

Replace (h,k) with (x,y)

Thus, the locus of a point is $x^2 + y^2 = 4$ ans

9. Question

If A (-1, 1) and B (2, 3) are two fixed points, find the locus of a point P so that the area of $\Delta PAB = 8$ sq. units.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by- $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

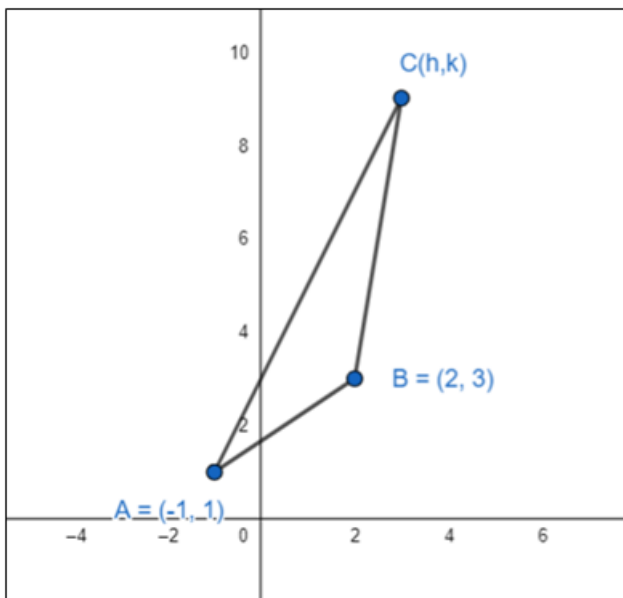
Area of a ΔPQR - Let $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be the 3 vertices of ΔPQR .

$$Ar(\Delta PQR) = \frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k). Name the moving point to be C

Given the area of $\Delta ABC = 8$



According to question:

$$8 = \frac{1}{2} |-1(3 - k) + 2(k - 1) + h((1 - 3))|$$

$$\Rightarrow 16 = |-3 + k + 2k - 2 + h - 3h|$$

$$\Rightarrow |3k - 2h - 5| = 16$$

$$\therefore 3k - 2h - 5 = 16 \text{ or } 3k - 2h - 5 = -16$$

$$\Rightarrow 3k - 2h - 21 = 0 \text{ or } 3k - 2h + 11 = 0$$

Replace (h, k) with (x, y)

Thus, locus of point is $3y - 2x - 21 = 0$ or $3y - 2x + 11 = 0$ ans

10. Question

A rod of length l slides between the two perpendicular lines. Find the locus of the point on the rod which divides it in the ratio 1 : 2.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by- **AB** = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

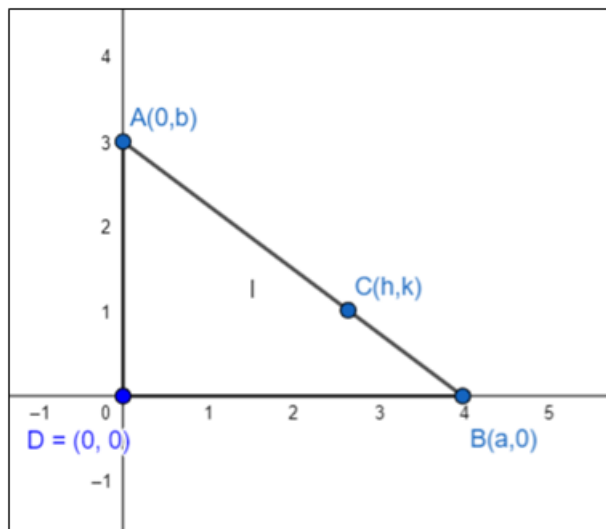
• **Idea of section formula-** Let two points $A(x_1, y_1)$ and $B(x_2, y_2)$ forms a line segment. If a point $C(x, y)$ divides line segment AB in the ratio of $m:n$ internally, then coordinates of C is given as:

$$C = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k) . Name the moving point to be C

Assume the two perpendicular lines on which rod slides are x and y-axis respectively.



Here line segment AB represents the rod of length l also $\triangle ADB$ formed is a right triangle. Coordinates of A and B are assumed to be $(0, b)$ and $(a, 0)$ respectively.

$$\therefore a^2 + b^2 = l^2 \dots \text{eqn 1}$$

As, (h, k) divides AB in ratio of 1:2

$\therefore$ from section formula we have coordinate of point C as-

$$C = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right) = \left(\frac{1 \times 0 + 2 \times a}{2+1}, \frac{1 \times b + 2 \times 0}{2+1} \right) = \left(\frac{2a}{3}, \frac{b}{3} \right)$$

As, a and b are assumed parameters so we have to remove it.

$$\because h = \frac{2a}{3} \Rightarrow a = \frac{3h}{2}$$

$$\text{And } k = \frac{b}{3} \Rightarrow b = 3k$$

From eqn 1:

$$a^2 + b^2 = l^2$$

$$\therefore \left(\frac{3h}{2} \right)^2 + (3k)^2 = l^2$$

$$\Rightarrow \frac{9h^2}{4} + 9k^2 = l^2 \Rightarrow \frac{h^2}{4} + k^2 = \frac{l^2}{9}$$

Replace (h, k) with (x, y)

$$\text{Thus, the locus of a point on the rod is: } \frac{x^2}{4} + y^2 = \frac{l^2}{9} \dots \text{ans}$$

11. Question

Find the locus of the mid-point of the portion of the $x \cos \alpha + y \sin \alpha = p$ which is intercepted between the

axes.

Answer

Key points to solve the problem:

• **Idea of distance formula-** Distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by- **$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$**

• **Idea of section formula-** Let two points $A(x_1, y_1)$ and $B(x_2, y_2)$ forms a line segment. If a point $C(x, y)$ divides line segment AB in the ratio of $m:n$ internally, then coordinates of C is given as:

$$C = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right) \text{ when } m = n = 1, C \text{ becomes the midpoint of } AB \text{ and } C \text{ is given as } C = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right)$$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k) . Name the moving point to be C

Given that (h, k) is the midpoint of line $x \cos \alpha + y \sin \alpha = p$ intercepted between axes.

So we need to find the points at which $x \cos \alpha + y \sin \alpha = p$ cuts the axes after which we will apply the section formula to get the locus.

Put $y = 0$

$\therefore x = p/\cos \alpha \Rightarrow$ coordinates on x -axis is $(p/\cos \alpha, 0)$. Name the point A

Similarly, Put $x = 0$

$\therefore y = p/\sin \alpha \Rightarrow$ coordinates on y -axis is $(0, p/\sin \alpha)$. Name this point B

As $C(h, k)$ is the midpoint of AB

$\therefore$ coordinate of C is given by:

$$C = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right) = \left(\frac{\frac{p}{\cos \alpha} + 0}{2}, \frac{0 + \frac{p}{\sin \alpha}}{2} \right) = \left(\frac{p}{2\cos \alpha}, \frac{p}{2\sin \alpha} \right)$$

Thus,

$$h = \frac{p}{2\cos \alpha} \Rightarrow \frac{p}{2h} = \cos \alpha \text{ ...equation 1}$$

$$\text{and } k = \frac{p}{2\sin \alpha} \Rightarrow \frac{p}{2k} = \sin \alpha \text{ ...equation 2}$$

Squaring and adding equation 1 and 2:

$$\frac{p^2}{4h^2} + \frac{p^2}{4k^2} = \cos^2 \alpha + \sin^2 \alpha$$

$$\Rightarrow \frac{p^2}{4h^2} + \frac{p^2}{4k^2} = 1$$

Replace (h, k) with (x, y)

Thus, the locus of a point on the rod is: $\frac{p^2}{4x^2} + \frac{p^2}{4y^2} = 1$ ans

12. Question

If O is the origin and Q is a variable point on $y^2 = x$, Find the locus of the mid-point of OQ .

Answer

Key points to solve the problem:

• **Idea of section formula-** Let two points $A(x_1, y_1)$ and $B(x_2, y_2)$ forms a line segment. If a point $C(x, y)$ divides line segment AB in the ratio of $m:n$ internally, then coordinates of C is given as:

$C = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right)$ when $m = n = 1$, C becomes the midpoint of AB and C is given as $C = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right)$

How to approach: To find the locus of a point we first assume the coordinate of the point to be (h, k) and write a mathematical equation as per the conditions mentioned in the question and finally replace (h, k) with (x, y) to get the locus of the point.

Let the coordinates of a point whose locus is to be determined to be (h, k). Name the moving point to be C

As, coordinate of mid point is (h,k) {by our assumption},

Let Q(a,b) be the point such that Q lies on curve $y^2 = x$

$b^2 = a$ **equation 1**

According to question C is the midpoint of OQ

$$\therefore C = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2} \right) \Rightarrow C = \left(\frac{a+0}{2}, \frac{b+0}{2} \right) = \left(\frac{a}{2}, \frac{b}{2} \right)$$

$$\therefore h = \frac{a}{2} \text{ or } a = 2h$$

$$\text{Similarly, } k = \frac{b}{2} \text{ or } b = 2k$$

Putting values of a and b in equation 1, we have:

$$(2k)^2 = 2h \Rightarrow 4k^2 = 2h \Rightarrow 2k^2 = h$$

Replace (h,k) with (x,y)

Thus, the locus of a point is: $2y^2 = x$ ans

Very Short Answer

1. Question

Write the sum of the series : $2 + 4 + 6 + 8 + \dots + 2n$

Answer

Let $S = 2 + 4 + 6 + 8 + \dots + 2n$

$S = 2(1 + 2 + 3 + 4 + \dots + n)$

$$\sum_{r=1}^n r = 1 + 2 + 3 + \dots + n = \left[\frac{n(n+1)}{2} \right]$$

Substituting the above value

$$S = 2 \left[\frac{n(n+1)}{2} \right]$$

$$S = n(n+1)$$

2. Question

Write the sum of the series : $1^2 - 2^2 + 3^2 - 4^2 + 5^2 - 6^2 + \dots + (2n-1)^2 - (2n)^2$

Answer

$$S = 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 - \{2^2 + 4^2 + 6^2 + \dots + (2n)^2\}$$

$$S = 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 - 2^2 \{1^2 + 2^2 + 3^2 + \dots + (n)^2\} \dots\dots(1)$$

We know

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right] \dots\dots(2)$$

$$\sum_{r=1}^{2n} r^2 = 1^2 + 2^2 + 3^2 \dots (2n-1)^2 + 4n^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right]$$

$$1^2 + 3^2 \dots (2n-1)^2 + 2^2 + 4^2 \dots + (2n)^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right]$$

$$1^2 + 3^2 \dots (2n-1)^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right] - 2^2 - 4^2 \dots - (2n)^2$$

$$1^2 + 3^2 \dots (2n-1)^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right] - 2^2 \{1^2 + 2^2 \dots + (n)^2\} \dots (3)$$

Substituting (3) in (1), we get,

$$s = \left[\frac{2n(2n+1)(4n+1)}{6} \right] - 2^2 \{1^2 + 2^2 \dots + (n)^2\} - 2^2 \{1^2 + 2^2 + \dots + (n)^2\}$$

$$s = \left[\frac{2n(2n+1)(4n+1)}{6} \right] - (2)2^2 \{1^2 + 2^2 \dots + (n)^2\}$$

Substituting (2) in the above equation

$$s = \left[\frac{2n(2n+1)(4n+1)}{6} \right] - (2)2^2 \left\{ \left[\frac{n(n+1)(2n+1)}{6} \right] \right\}$$

$$s = \left[\frac{2n(2n+1)(4n+1) - 8n(n+1)(2n+1)}{6} \right]$$

3. Question

Write the sum to n terms of a series whose rth term is: $r + 2^r$

Answer

The general term be $= i + 2^i$

$$\sum_{i=1}^n i + 2^i = \text{Sum of series}$$

We know by property that:

$$\sum a x^n + b x^{n-1} + c x^{n-2} \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots + d_0 \sum 1$$

Thus

$$\sum_{i=1}^n i + 2^i = \sum_{i=1}^n i + \sum_{i=1}^n 2^i$$

$$\sum_{i=1}^n i + 2^i = \sum_{i=1}^n i + 2(2^n - 1)$$

$$\sum_{n=1}^n n = 1 + 2 + 3 \dots + n = \left[\frac{n(n+1)}{2} \right]$$

Substituting the above value

Thus

$$\sum_{i=1}^n i + 2i = \left[\frac{n(n+1)}{2} \right] + 2(2^n - 1)$$

4. Question

If $\sum_{r=1}^n r = 55$, find $\sum_{r=1}^n r^3$.

Answer

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots n = \frac{n(n+1)}{2} \dots\dots\dots (1)$$

Given

$$\sum_{r=1}^n r = 55$$

From (1) we have

$$\frac{n(n+1)}{2} = 55$$

Solving the above equation

$$n = 10$$

We know

$$\sum_{r=1}^n r^3 = 1^3 + 2^3 + 3^3 \dots\dots\dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2 \dots\dots\dots (2)$$

Thus

Putting $n = 10$ in eq(2)

$$= \left[\frac{10(10+1)}{2} \right]^2$$

$$= 55^2$$

$$= 3025$$

5. Question

If the sum of first n even natural numbers is equal to k times the sum of first n odd natural numbers, then write the value of k .

Answer

we know

$$\sum_{r=1}^{2n} r = 1 + 2 + 3 \dots 2n = \frac{2n(2n+1)}{2}$$

$$\sum_{r=1}^n 2r = 2 + 4 + 6 \dots 2n = \frac{2n(n+1)}{2} \dots\dots\dots (1)$$

$$\text{Given } 2 + 4 + 6 + 8 \dots 2n = k(1 + 3 + 5 \dots 2n - 1)$$

$$\text{From (1) } n(n+1) = k(1 + 3 + 5 \dots 2n - 1) \dots\dots\dots (2)$$

$$1 + 3 + 5 \dots (2n-1) + 2 + 4 + \dots 2n = \frac{2n(2n+1)}{2} \dots\dots\dots (3)$$

Thus substituting the values from (1) in (3), we get,

$$1 + 3 + 5 \dots (2n-1) + \frac{2n(n+1)}{2} = \frac{2n(2n+1)}{2}$$

$$1 + 3 + 5 \dots (2n-1) = n(2n+1) - n(n+1)$$

$$1 + 3 + 5 \dots (2n-1) = n^2 \dots\dots\dots (4)$$

Substituting (4) in (2), we get,

$$n(n + 1) = k(n^2)$$

$$k = (n + 1)/n$$

6. Question

Write the sum of 20 terms of the series :

$$1 + \frac{1}{2}(1 + 2) + \frac{1}{3}(1 + 2 + 3) + \dots$$

Answer

The general term would be

$$\frac{r + (r - 1) + (r - 2) + \dots + (r - (r - 1))}{r}$$

$$= \frac{r \times r - (1 + 2 + 3 + \dots + r - 1)}{r} \dots \dots \dots (1)$$

$$\text{Since, } \sum_{i=1}^r i = 1 + 2 + 3 + \dots + r = \left[\frac{r(r + 1)}{2} \right]$$

$$\sum_{i=1}^{r-1} i = 1 + 2 + 3 + \dots + r - 1 = \left[\frac{r(r - 1)}{2} \right]$$

From equation (1), we get,

$$= \frac{r \times r - \frac{r(r - 1)}{2}}{r}$$

$$r - \frac{(r - 1)}{2} = \frac{(r + 1)}{2}$$

Thus the general term would be, $\frac{(r + 1)}{2}$

$$\text{To find } \sum_{r=1}^{20} \frac{(r + 1)}{2} \quad (2)$$

We know by property that:

$$\sum a_n + b_n + c_n + \dots + d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} + \dots + d_0 \sum 1$$

Since ,

$$\sum_{r=1}^n r = 1 + 2 + 3 + \dots + n = \left[\frac{n(n + 1)}{2} \right]$$

$$\sum_{r=1}^n 1 = 1 + 1 + 1 + \dots + 1 = [n]$$

Thus equation (2) becomes

$$\sum_{r=1}^{20} \frac{(r + 1)}{2} = \frac{1}{2} \sum_{r=1}^{20} r + \frac{1}{2} \sum_{r=1}^{20} 1$$

$$\sum_{r=1}^{20} \frac{(r + 1)}{2} = \frac{1}{2} \left[\frac{n(n + 1)}{2} \right] + \frac{n}{2} = 95$$

7. Question

Write the 50th term of the series $2 + 3 + 6 + 11 + 18 + \dots$

Answer

Let $s = 2 + 3 + 6 + 11 + 18 + \dots + n$

By shifting each term by one

$$S = 2 + 3 + 6 + 11 + 18 + \dots + \text{nth} \dots \dots \dots (1)$$

$$S = 2 + 3 + 6 + 11 + 18 + \dots + (n - 1)\text{th} + \text{nth} \dots (2)$$

by (1) - (2) we get

$$0 = 2 + 1 + 3 + 5 + 7 + \dots + \text{nth} - (n - 1)\text{th} - \text{nth}$$

$$N\text{th} = 2 + (1 + 3 + 5 + 7 + 9 + \dots + 2r - 1) \dots \dots (3)$$

$$N\text{th} = 2 + (\text{summation of first } (n - 1)\text{th term})$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots \dots + d_0 \sum 1$$

Therefore,

$$\sum_{r=1}^{n-1} 2r - 1 = 2 \sum_{r=1}^{n-1} r - \sum_{r=1}^{n-1} 1 = 1 + 3 + 5 \dots = [n - 1]^2$$

Since,

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots + n = \left[\frac{n(n + 1)}{2} \right]$$

$$\sum_{r=1}^{n-1} r = 1 + 2 + 3 \dots + n = \left[\frac{n(n - 1)}{2} \right]$$

$$\sum_{r=1}^n 1 = 1 + 1 + 1 \dots + 1 = [n]$$

$$\sum_{r=1}^{n-1} 1 = 1 + 1 + 1 \dots + 1 = [n - 1]$$

Thus from (3)

$$N\text{th} = 2 + (n - 1)^2$$

Hence 50th term be

$$50^{\text{th}} = 2 + (50 - 1)^2$$

$$50^{\text{th}} = 2 + (49)^2$$

8. Question

Let S_n denote the sum of the cubes of first n natural numbers, and s_n denote the sum of first n natural

numbers. Then write the value of $\sum_{r=1}^n \frac{S_r}{s_r}$

Answer

To find

Let $I = \sum_{r=1}^n \frac{S_r}{s_r} \dots\dots (1)$

Given,

$$S_r = \sum_{r=1}^n r^3 = 1^3 + 2^3 + 3^3 \dots\dots\dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

$$s_r = \sum_{r=1}^n r = 1 + 2 + 3 \dots\dots\dots + n = \left[\frac{n(n+1)}{2} \right]$$

Substituting in equation (1)

$$I = \sum_{r=1}^n \frac{\left[\frac{n(n+1)}{2} \right]^2}{\left[\frac{n(n+1)}{2} \right]}$$

$$I = \sum_{r=1}^n \left[\frac{n(n+1)}{2} \right]$$

$$I = \sum_{r=1}^n \frac{[n^2 + n]}{2}$$

We know by property that:

$$\sum ax^n + bx^{n-1} + cx^{n-2} \dots\dots\dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots\dots\dots + d_0 \sum 1$$

Thus

$$I = \frac{1}{2} \sum_{r=1}^n n^2 + \frac{1}{2} \sum_{r=1}^n n$$

And We know

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots\dots\dots + n = \left[\frac{n(n+1)}{2} \right]$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots\dots\dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

Substituting the values

$$I = \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} \right] + \frac{1}{2} \left[\frac{n(n+1)}{2} \right]$$

$$I = \frac{n(n+1)}{4} \left\{ \left[\frac{(2n+1)}{3} \right] + [1] \right\}$$

$$I = \frac{n(n+1)}{4} \left\{ \left[\frac{(2n+4)}{3} \right] \right\}$$

$$I = \frac{n(n+1)}{2} \left\{ \left[\frac{(n+2)}{3} \right] \right\}$$

MCQ

1. Question

The sum to n terms of the series $\frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{7}} + \dots\dots\dots$ is

- A. $\sqrt{2n+1}$
- B. $\frac{1}{2}\sqrt{2n+1}$
- C. $\sqrt{2n+1}-1$
- D. $\frac{1}{2}\{\sqrt{2n+1}-1\}$

Answer

To find:

$$\frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{7}} \dots \frac{1}{\sqrt{2n-1} + \sqrt{2n+1}}$$

Rationalizing the above equation:

$$\begin{aligned} &= \frac{1(-\sqrt{1} + \sqrt{3})}{(\sqrt{1} + \sqrt{3})(-\sqrt{1} + \sqrt{3})} + \frac{1(-\sqrt{3} + \sqrt{5})}{(\sqrt{3} + \sqrt{5})(-\sqrt{3} + \sqrt{5})} \\ &+ \frac{1(-\sqrt{5} + \sqrt{7})}{(\sqrt{5} + \sqrt{7})(-\sqrt{5} + \sqrt{7})} \dots \frac{1}{(\sqrt{2n-1} + \sqrt{2n+1})} \left\{ \frac{(-\sqrt{2n-1} + \sqrt{2n+1})}{(-\sqrt{2n-1} + \sqrt{2n+1})} \right\} \\ &= \frac{1(-\sqrt{1} + \sqrt{3})}{3-1} + \frac{1(-\sqrt{3} + \sqrt{5})}{5-3} \\ &\quad + \frac{1(-\sqrt{5} + \sqrt{7})}{7-5} \dots \left\{ \frac{(-\sqrt{2n-1} + \sqrt{2n+1})}{2n+1-(2n-1)} \right\} \end{aligned}$$

since $a^2 - b^2 = (a + b)(a - b)$

$$\begin{aligned} &= \frac{1}{2} [(-\sqrt{1} + \sqrt{3}) + (-\sqrt{3} + \sqrt{5}) + (-\sqrt{5} + \sqrt{7}) + \dots + (-\sqrt{2n-1} \\ &\quad + \sqrt{2n+1})] \\ &= \frac{1}{2} [-1 + \sqrt{2n+1}] \end{aligned}$$

2. Question

The sum of the series : $\frac{1}{\log_2 4} + \frac{1}{\log_4 4} + \frac{1}{\log_8 4} + \dots + \frac{1}{\log_{2^n} 4}$ is

- A. $\frac{n(n+1)}{2}$
- B. $\frac{n(n+1)(2n+1)}{12}$
- C. $\frac{n(n+1)}{4}$

D. None of these

Answer

$$\frac{1}{\log_2 4} + \frac{1}{\log_4 4} + \frac{1}{\log_8 4} \dots \dots \dots \frac{1}{\log_{2^n} 4}$$

We know $\log_a b = \log b / \log a$

$$\frac{\log 2}{\log 4} + \frac{\log 4}{\log 4} + \frac{\log 8}{\log 4} \dots \dots \dots \frac{\log 2^n}{\log 4}$$

We know $\log m^n = n \log m$

$$= \frac{\log 2}{\log 4} + \frac{\log 2^2}{\log 4} + \frac{\log 2^3}{\log 4} \dots \dots \dots \frac{\log 2^n}{\log 4}$$

$$= \frac{\log 2}{\log 4} + \frac{2 \log 2}{\log 4} + \frac{3 \log 2}{\log 4} \dots \dots \dots \frac{n \log 2}{\log 4}$$

$$= \frac{\log 2 [1 + 2 + 3 \dots n]}{\log 4}$$

$$= \frac{\log 2 [1 + 2 + 3 \dots n]}{2 \log 2}$$

$$= [1 + 2 + 3 \dots n] / 2$$

$$= n(n + 1) / 4$$

3. Question

The value of $\sum_{r=1}^n \left\{ (2r-1)a + \frac{1}{b^r} \right\}$ is equal to

A. $an^2 + \frac{b^{n-1} - 1}{b^{n-1}(b-1)}$

B. $an^2 + \frac{b^n - 1}{b^n(b-1)}$

C. $an^3 + \frac{b^{n-1} - 1}{b^n(b-1)}$

D. none of these

Answer

We know by property that $\sum ax^n + bx^{n-1} + cx^{n-2} \dots \dots \dots d_0 = a \sum x^n + b \sum x^{n-1} + c \sum x^{n-2} \dots \dots \dots + d_0 \sum 1$

$$\sum_{r=1}^n (2r-1)a + \frac{1}{b^r} = 2a \sum_{r=1}^n r - a \sum_{r=1}^n 1 + \sum_{r=1}^n \frac{1}{b^r}$$

We know

$$\sum_{r=1}^n r = 1 + 2 + 3 \dots \dots \dots + n = \left[\frac{n(n+1)}{2} \right]$$

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots \dots \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right]$$

$$\sum_{r=1}^n 1 = 1 + 1 + \dots \dots \dots n \text{ times} = n$$

$$2a \sum_{r=1}^n r = \frac{2an(n+1)}{2} \quad (1)$$

$$a \sum_{r=1}^n 1 = an \quad (2)$$

$$\sum_{r=1}^n \frac{1}{b^r} = \frac{1/b \left[1 - \left(\frac{1}{b} \right)^n \right]}{\left[1 - \frac{1}{b} \right]}$$

$$\sum_{r=1}^n \frac{1}{b^r} = \frac{1 \left[1 - \left(\frac{1}{b} \right)^n \right] b}{b[b-1]}$$

$$\sum_{r=1}^n \frac{1}{b^r} = \frac{\left[1 - \left(\frac{1}{b} \right)^n \right]}{[b-1]}$$

$$\sum_{r=1}^n \frac{1}{b^r} = \frac{\left[1 - \left(\frac{1}{b} \right)^n \right]}{[b-1]}$$

$$\sum_{r=1}^n \frac{1}{b^r} = \frac{\left[1 - \left(\frac{1}{b} \right)^n \right]}{[b-1]}$$

$$\sum_{r=1}^n \frac{1}{b^r} = \frac{1 - \left(\frac{1}{b} \right)^n}{[b-1]}$$

$$\sum_{r=1}^n \frac{1}{b^r} = \frac{b^n - 1}{b^n[b-1]} \quad (3)$$

Adding (1) (2) and (3)

$$\begin{aligned} \sum_{r=1}^n (2r-1)a + \frac{1}{b^r} &= 2a \sum_{r=1}^n r - a \sum_{r=1}^n 1 + \sum_{r=1}^n \frac{1}{b^r} \\ &= \frac{2an(n+1)}{2} - an + \frac{b^n - 1}{b^n[b-1]} \\ &= \frac{b^n - 1}{b^n[b-1]} + an^2 \end{aligned}$$

4. Question

If $\sum n = 210$, then $\sum n^2 =$

- A. 2870
- B. 2160
- C. 2970
- D. none of these

Answer

$$\sum n = \frac{n(n+1)}{2} = 210$$

Solving we get $n=20$

$$\sum n^2 = \frac{n(n+1)(2n+1)}{6}$$

Substituting the value

$$\frac{20(20+1)(40+1)}{6}$$

$$10(7)(41)$$

$$2870$$

5. Question

If $S_n = \sum_{r=1}^n \frac{1+2+2^2+\dots+\text{Sum to } r \text{ terms}}{2^r}$, then S_n is equal to

A. $2^n - n - 1$

B. $1 - \frac{1}{2^n}$

C. $n - 1 + \frac{1}{2^n}$

D. $2^n - 1$

Answer

$$1 + 2 + 2^2 + \dots + 2^{r-1} = 1(2^r - 1)/2 - 1$$

$$1 + 2 + 2^2 + \dots + 2^{r-1} = 1(2^r - 1)$$

$$\sum_{r=1}^n \frac{(2^r - 1)}{2^r}$$

$$\sum_{r=1}^n 1 - \frac{1}{2^r}$$

$$\sum_{r=1}^n 1 - \sum_{r=1}^n \frac{1}{2^r}$$

$$n - \frac{1}{2} \left[\frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} \right]$$

$$n - 1 + \frac{1}{2^n}$$

6. Question

If $1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \dots$ to n terms is S . Then, S is equal to

A. $\frac{n(n+3)}{4}$

B. $\frac{n(n+2)}{4}$

C. $\frac{n(n+1)(n+2)}{6}$

D. n^2

Answer

$$1 + \frac{1+2}{2} + \frac{1+2+3}{3} + \dots + \frac{1+2+3+\dots+n}{n}$$

$$\frac{1+2+3+\dots+n}{n} = \frac{n(n+1)}{2n}$$

$$\frac{1+2+3+\dots+n}{n} = \frac{(n+1)}{2}$$

Thus the nth term would be

$$\text{nth} = (n+1)/2$$

the general term would be

$$\text{rth} = (r+1)/2$$

$$\sum_{r=1}^n \frac{r+1}{2}$$

We know by property that $\sum ax^n + bx^{n-1} + cx^{n-2} + \dots + d_0 = a\sum x^n + b\sum x^{n-1} + c\sum x^{n-2} + \dots + d_0\sum 1$

$$\sum_{r=1}^n \frac{r+1}{2} = 1/2 \left[\sum_{r=1}^n r + \sum_{r=1}^n 1 \right]$$

$$= \frac{1}{2} \left[\frac{n(n+1)}{2} + n \right]$$

$$= \frac{1}{2} \left[\frac{n(n+1) + 2n}{2} \right]$$

$$= \frac{1}{2} \left[\frac{n^2 + 3n}{2} \right]$$

$$= \left[\frac{n(n+3)}{4} \right]$$

7. Question

Sum of n terms of the series $\sqrt{2} + \sqrt{8} + \sqrt{18} + \sqrt{32} + \dots$ is

A. $\frac{n(n+1)}{2}$

B. $2n(n+1)$

C. $\frac{n(n+1)}{\sqrt{2}}$

D. 1

Answer

$$\text{Let } S = \sqrt{2} + \sqrt{8} + \sqrt{18} + \sqrt{32} + \dots$$

It can be written as,

$$S = \sqrt{2}(1 + 2 + 3 + \dots + n)$$

We know that,

$$1 + 2 + 3 + 4 + \dots + n = \frac{n(n+1)}{2}$$

$$S = \frac{\sqrt{2}(n(n+1))}{2}$$

$$S = \frac{n(n+1)}{\sqrt{2}}$$

8. Question

The sum of 10 terms of the series $\sqrt{2} + \sqrt{6} + \sqrt{18} + \dots$ is

A. $121(\sqrt{6} + \sqrt{2})$

B. $243(\sqrt{3} + 1)$

C. $\frac{121}{\sqrt{3}-1}$

D. $242(\sqrt{3} - 1)$

Answer

Let $S = \sqrt{2} + \sqrt{6} + \sqrt{18} + \dots$

It can also be written as,

$$S = \sqrt{2}(1 + \sqrt{3} + 3 + 3\sqrt{3} + \dots)$$

Now,

$1 + \sqrt{3} + 3 + 3\sqrt{3} + \dots$ is a G.P with common ratio $\sqrt{3}$

Sum of the 10 terms will be given by,

$$S_{10} = \sqrt{2} \left(1 \frac{(\sqrt{3})^{10} - 1}{\sqrt{3} - 1} \right)$$

$$S_{10} = \sqrt{2} \left(\frac{3^5 - 1}{\sqrt{3} - 1} \right)$$

$$S_{10} = \sqrt{2} \left(\frac{243 - 1}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} \right)$$

$$S_{10} = \sqrt{2}(121)(\sqrt{3} + 1)$$

$$S_{10} = 121(\sqrt{6} + \sqrt{2})$$

9. Question

The sum of the series $1^2 + 3^2 + 5^2 + \dots$ to n terms is

A. $\frac{n(n+1)(2n+1)}{2}$

B. $\frac{n(2n-1)(2n+1)}{3}$

$$C. \frac{(n-1)^2(2n-1)}{6}$$

$$D. \frac{(2n+1)^3}{3}$$

Answer

We know

$$\sum_{r=1}^n r^2 = 1^2 + 2^2 + 3^2 \dots \dots \dots + n^2 = \left[\frac{n(n+1)(2n+1)}{6} \right] \dots \dots \dots (1)$$

$$\sum_{r=1}^{2n} r^2 = 1^2 + 2^2 + 3^2 \dots (2n-1)^2 + 4n^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right]$$

$$S = 1^2 + 3^2 \dots (2n-1)^2 + 2^2 + 4^2 \dots + (2n)^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right]$$

$$1^2 + 3^2 \dots (2n-1)^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right] - 2^2 - 4^2 \dots - (2n)^2$$

$$1^2 + 3^2 \dots (2n-1)^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right] - 2^2 \{1^2 + 2^2 \dots + (n)^2\} \dots (2)$$

Substituting (2) in (1)

$$1^2 + 3^2 \dots (2n-1)^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right] - 2^2 \{1^2 + 2^2 \dots + (n)^2\}$$

$$1^2 + 3^2 \dots (2n-1)^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right] - 2^2 \left\{ \frac{n(n+1)(2n+1)}{6} \right\}$$

$$1^2 + 3^2 \dots (2n-1)^2 = \left[\frac{2n(2n+1)(4n+1)}{6} \right] - 2^2 \left\{ \left[\frac{n(n+1)(2n+1)}{6} \right] \right\}$$

$$1^2 + 3^2 \dots (2n-1)^2 = \left[\frac{2n(2n+1)(4n+1) - 4n(n+1)(2n+1)}{6} \right]$$

$$1^2 + 3^2 \dots (2n-1)^2 = \left[\frac{n(2n+1)(2n-1)}{3} \right]$$

10. Question

The sum of the series $\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} + \dots$ to n terms is

$$A. n - \frac{1}{2}(3^{-n} - 1)$$

$$B. n - \frac{1}{2}(1 - 3^{-n})$$

$$C. n + \frac{1}{2}(3^n - 1)$$

$$D. n - \frac{1}{2}(3^n - 1)$$

Answer

We can write

$$\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} \dots = \frac{3-1}{3} + \frac{9-1}{9} + \frac{27-1}{27} + \frac{81-1}{81} \dots$$

$$\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} \dots = 1 + \frac{-1}{3} + 1 + \frac{-1}{9} + 1 + \frac{-1}{27} + 1 + \frac{-1}{81} \dots$$

$$\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} \dots = n - \left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \dots \frac{1}{3^n} \right)$$

$$\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} \dots = n - \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} \dots \frac{1}{3^n} \right)$$

Since

$$\sum_{r=1}^n \frac{1}{b^r} = \frac{1/b \left[1 - \left(\frac{1}{b} \right)^n \right]}{\left[1 - \frac{1}{b} \right]}$$

$$\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} \dots = n - \frac{1}{3} \left[\frac{1 - \frac{1}{3^n}}{1 - \frac{1}{3}} \right]$$

$$\frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{80}{81} \dots = n - \left[\frac{1 - \frac{1}{3^n}}{2} \right]$$