

15. Mean Value Theorems

Exercise 15.1

1 A. Question

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals :

$$f(x) = 3 + (x - 2)^{2/3} \text{ on } [1, 3]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

(i) Given function is:

$$\Rightarrow f(x) = 3 + (x - 2)^{\frac{2}{3}} \text{ on } [1, 3]$$

Let us check the differentiability of the function f(x).

Let's find the derivative of f(x),

$$\Rightarrow f'(x) = \frac{d}{dx} \left(3 + (x - 2)^{\frac{2}{3}} \right)$$

$$\Rightarrow f'(x) = \frac{d(3)}{dx} + \frac{d\left((x-2)^{\frac{2}{3}}\right)}{dx}$$

$$\Rightarrow f'(x) = 0 + \frac{2}{3} (x - 2)^{\frac{2}{3} - 1}$$

$$\Rightarrow f'(x) = \frac{2}{3} (x - 2)^{-\frac{1}{3}}$$

$$\Rightarrow f'(x) = \frac{2}{3(x-2)^{\frac{1}{3}}}$$

Let's the differentiability at the value of $x = 2$

$$\Rightarrow \lim_{x \rightarrow 2} f'(x) = \lim_{x \rightarrow 2} \frac{2}{3(x-2)^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x \rightarrow 2} f'(x) = \frac{2}{3(2-2)^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x \rightarrow 2} f'(x) = \frac{2}{3(0)}$$

$$\Rightarrow \lim_{x \rightarrow 2} f'(x) = \text{undefined}$$

$\therefore$ f is not differentiable at $x = 2$, so it is not differentiable in the closed interval (1,3).

So, Rolle's theorem is not applicable for the function f on the interval [1,3].

1 B. Question

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals :

$f(x) = [x]$ for $-1 \leq x \leq 1$, where $[x]$ denotes the greatest integer not exceeding x

Answer

Given function is:

$\Rightarrow f(x) = [x], -1 \leq x \leq 1$ where $[x]$ denotes the greatest integer not exceeding x .

Let us check the continuity of the function 'f'.

Here in the interval $x \in [-1, 1]$, the function has to be Right continuous at $x = 1$ and left continuous at $x = -1$.

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} [x]$$

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1+h} [x] \text{ where } h > 0.$$

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} 1$$

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = 1 \dots\dots(1)$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} [x]$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1-h} [x], \text{ where } h > 0$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} 0$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = 0 \dots\dots(2)$$

From (1) and (2), we can see that the limits are not the same so, the function is not continuous in the interval $[-1, 1]$.

$\therefore$ Rolle's theorem is not applicable for the function f in the interval $[-1, 1]$.

1 C. Question

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals :

$f(x) = \sin 1/x$ for $-1 \leq x \leq 1$

Answer

Given function is:

$$\Rightarrow f(x) = \sin\left(\frac{1}{x}\right) \text{ for } -1 \leq x \leq 1$$

Let us check the continuity of the function 'f' at the value of $x = 0$.

We can not directly find the value of limit at $x = 0$, as the function is not valid at $x = 0$. So, we take the limit on either sides and $x = 0$, and we check whether they are equal or not.

Right - Hand Limit:

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sin\left(\frac{1}{x}\right)$$

We assume that the limit $\lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right) = k, k \in [-1, 1]$.

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0+h} \sin\left(\frac{1}{x}\right), \text{ where } h > 0$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \sin\left(\frac{1}{h+0}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^+} f(x) = k \dots\dots(1)$$

Left - Hand Limit:

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \sin\left(\frac{1}{x}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^- h} \sin\left(\frac{1}{x}\right), \text{ where } h > 0$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} \sin\left(\frac{1}{0-h}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} \sin\left(\frac{1}{-h}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} -\sin\left(\frac{1}{h}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = -\lim_{h \rightarrow 0} \sin\left(\frac{1}{h}\right)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = -k \dots\dots(2)$$

From (1) and (2), we can see that the Right hand and left - hand limits are not equal, so the function 'f' is not continuous at $x = 0$.

$\therefore$ Rolle's theorem is not applicable to the function 'f' in the interval $[-1, 1]$.

1 D. Question

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals :

$$f(x) = 2x^2 - 5x + 3 \text{ on } [1, 3]$$

Answer

Given function is:

$$\Rightarrow f(x) = 2x^2 - 5x + 3 \text{ on } [1, 3]$$

Since given function 'f' is a polynomial. So, it is continuous and differentiable every where.

Now, we find the values of function at the extremum values.

$$\Rightarrow f(1) = 2(1)^2 - 5(1) + 3$$

$$\Rightarrow f(1) = 2 - 5 + 3$$

$$\Rightarrow f(1) = 0 \dots\dots(1)$$

$$\Rightarrow f(3) = 2(3)^2 - 5(3) + 3$$

$$\Rightarrow f(3) = 2(9) - 15 + 3$$

$$\Rightarrow f(3) = 18 - 12$$

$$\Rightarrow f(3) = 6 \dots\dots(2)$$

From (1) and (2), we can say that,

$$f(1) \neq f(3)$$

$\therefore$ Rolle's theorem is not applicable for the function f in interval $[1, 3]$.

1 E. Question

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals :

$$f(x) = x^{2/3} \text{ on } [-1, 1]$$

Answer

Given function is:

$$\Rightarrow f(x) = x^{\frac{2}{3}} \text{ on } [-1, 1]$$

Let's find the derivative of the given function:

$$\Rightarrow f'(x) = \frac{d\left(x^{\frac{2}{3}}\right)}{dx}$$

$$\Rightarrow f'(x) = \frac{2}{3}x^{\frac{2}{3}-1}$$

$$\Rightarrow f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

$$\Rightarrow f'(x) = \frac{2}{3x^{\frac{1}{3}}}$$

Let's check the differentiability of the function at $x = 0$.

$$\Rightarrow \lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \frac{2}{3x^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x \rightarrow 0} f'(x) = \frac{2}{3(0)^{\frac{1}{3}}}$$

$$\Rightarrow \lim_{x \rightarrow 0} f'(x) = \text{undefined}$$

Since the limit for the derivative is undefined at $x = 0$, we can say that f is not differentiable at $x = 0$.

$\therefore$ Rolle's theorem is not applicable to the function ' f ' on $[-1, 1]$.

1 F. Question

Discuss the applicability of Rolle's theorem for the following functions on the indicated intervals :

$$f(x) = \begin{cases} -4x + 5, & 0 \leq x \leq 1 \\ 2x - 3, & 1 < x \leq 2 \end{cases}$$

Answer

Given function is:

$$\Rightarrow f(x) = \begin{cases} -4x + 5, & 0 \leq x \leq 1 \\ 2x - 3, & 1 < x \leq 2 \end{cases}$$

Let's check the continuity at $x = 1$ as the equation of function changes.

Left - Hand Limit:

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} -4x + 5$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = -4(1) + 5$$

$$\Rightarrow \lim_{x \rightarrow 1^-} f(x) = 1 \dots\dots(1)$$

Right - Hand Limit:

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2x - 3$$

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = 2(1) - 3$$

$$\Rightarrow \lim_{x \rightarrow 1^+} f(x) = -1 \dots\dots(2)$$

From (1) and (2), we can see that the values of both side limits are not equal. So, the function ' f ' is not continuous at $x = 1$.

$\therefore$ Rolle's theorem is not applicable to the function ' f ' in the interval $[0, 2]$.

2 A. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = x^2 - 8x + 12 \text{ on } [2, 6]$$

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = x^2 - 8x + 12 \text{ on } [2,6]$$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.

Let us find the values at extremums:

$$\Rightarrow f(2) = 2^2 - 8(2) + 12$$

$$\Rightarrow f(2) = 4 - 16 + 12$$

$$\Rightarrow f(2) = 0$$

$$\Rightarrow f(6) = 6^2 - 8(6) + 12$$

$$\Rightarrow f(6) = 36 - 48 + 12$$

$$\Rightarrow f(6) = 0$$

$\therefore f(2) = f(6)$, Rolle's theorem applicable for function 'f' on [2,6].

Let's find the derivative of f(x):

$$\Rightarrow f'(x) = \frac{d(x^2 - 8x + 12)}{dx}$$

$$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(8x)}{dx} + \frac{d(12)}{dx}$$

$$\Rightarrow f'(x) = 2x - 8 + 0$$

$$\Rightarrow f'(x) = 2x - 8$$

We have $f'(c) = 0$ $c \in (2,6)$, from the definition given above.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c - 8 = 0$$

$$\Rightarrow 2c = 8$$

$$\Rightarrow c = \frac{8}{2}$$

$$\Rightarrow c = 4 \in (2,6)$$

$\therefore$ Rolle's theorem is verified.

2 B. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = x^2 - 4x + 3 \text{ on } [1, 3]$$

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = x^2 - 4x + 3 \text{ on } [1,3]$$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.

Let us find the values at extremums:

$$\Rightarrow f(1) = 1^2 - 4(1) + 3$$

$$\Rightarrow f(1) = 1 - 4 + 3$$

$$\Rightarrow f(1) = 0$$

$$\Rightarrow f(3) = 3^2 - 4(3) + 3$$

$$\Rightarrow f(3) = 9 - 12 + 3$$

$$\Rightarrow f(3) = 0$$

$\therefore f(1) = f(3)$, Rolle's theorem applicable for function 'f' on [1,3].

Let's find the derivative of f(x):

$$\Rightarrow f'(x) = \frac{d(x^2 - 4x + 3)}{dx}$$

$$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(4x)}{dx} + \frac{d(3)}{dx}$$

$$\Rightarrow f'(x) = 2x - 4 + 0$$

$$\Rightarrow f'(x) = 2x - 4$$

We have $f'(c) = 0$ $c \in (1,3)$, from the definition given above.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c - 4 = 0$$

$$\Rightarrow 2c = 4$$

$$\Rightarrow c = \frac{4}{2}$$

$$\Rightarrow c = 2 \in (1,3)$$

$\therefore$ Rolle's theorem is verified.

2 C. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = (x - 1)(x - 2)^2 \text{ on } [1, 2]$$

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = (x - 1)(x - 2)^2 \text{ on } [1,2]$$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e, on R.

Let us find the values at extremums:

$$\Rightarrow f(1) = (1 - 1)(1 - 2)^2$$

$$\Rightarrow f(1) = 0(1)^2$$

$$\Rightarrow f(1) = 0$$

$$\Rightarrow f(2) = (2 - 1)(2 - 2)^2$$

$$\Rightarrow f(2) = 1.0^2$$

$$\Rightarrow f(2) = 0$$

$\therefore f(1) = f(2)$, Rolle's theorem applicable for function 'f' on [1,2].

Let's find the derivative of f(x):

$$\Rightarrow f'(x) = \frac{d((x-1)(x-2)^2)}{dx}$$

Differentiating using UV rule

$$\Rightarrow f'(x) = (x-2)^2 \times \frac{d(x-1)}{dx} + (x-1) \times \frac{d((x-2)^2)}{dx}$$

$$\Rightarrow f'(x) = ((x-2)^2 \times 1) + ((x-1) \times 2 \times (x-2))$$

$$\Rightarrow f'(x) = x^2 - 4x + 4 + 2(x^2 - 3x + 2)$$

$$\Rightarrow f'(x) = 3x^2 - 10x + 8$$

We have $f'(c) = 0$ $c \in (1,2)$, from the definition given above.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 3c^2 - 10c + 8 = 0$$

$$\Rightarrow c = \frac{10 \pm \sqrt{(-10)^2 - (4 \times 3 \times 8)}}{2 \times 3}$$

$$\Rightarrow c = \frac{10 \pm \sqrt{100 - 96}}{6}$$

$$\Rightarrow c = \frac{10 \pm 2}{6}$$

$$\Rightarrow c = \frac{12}{6} \text{ OR } c = \frac{8}{6}$$

$$\Rightarrow c = \frac{4}{3} \in (1,2) \text{ (neglecting the value 2)}$$

$\therefore$ Rolle's theorem is verified.

2 D. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = x(x - 1)^2 \text{ on } [0, 1]$$

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = x(x - 1)^2 \text{ on } [0,1]$$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e, on R.

Let us find the values at extremums:

$$\Rightarrow f(0) = 0(0 - 1)^2$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(1) = 1(1 - 1)^2$$

$$\Rightarrow f(1) = 0^2$$

$$\Rightarrow f(1) = 0$$

$\therefore f(0) = f(1)$, Rolle's theorem applicable for function 'f' on [0,1].

Let's find the derivative of f(x):

$$\Rightarrow f'(x) = \frac{d(x(x-1)^2)}{dx}$$

Differentiating using UV rule,

$$\Rightarrow f'(x) = (x - 1)^2 \times \frac{d(x)}{dx} + x \frac{d((x-1)^2)}{dx}$$

$$\Rightarrow f'(x) = ((x - 1)^2 \times 1) + (x \times 2 \times (x - 1))$$

$$\Rightarrow f'(x) = (x - 1)^2 + 2(x^2 - x)$$

$$\Rightarrow f'(x) = x^2 - 2x + 1 + 2x^2 - 2x$$

$$\Rightarrow f'(x) = 3x^2 - 4x + 1$$

We have $f'(c) = 0$ $c \in (0,1)$, from the definition given above.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 3c^2 - 4c + 1 = 0$$

$$\Rightarrow c = \frac{4 \pm \sqrt{(-4)^2 - (4 \times 3 \times 1)}}{2 \times 3}$$

$$\Rightarrow c = \frac{4 \pm \sqrt{16 - 12}}{6}$$

$$\Rightarrow c = \frac{4 \pm \sqrt{4}}{6}$$

$$\Rightarrow c = \frac{6}{6} \text{ or } c = \frac{2}{6}$$

$$\Rightarrow c = \frac{1}{3} \in (0,1)$$

$\therefore$ Rolle's theorem is verified.

2 E. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = (x^2 - 1)(x - 2) \text{ on } [-1, 2]$$

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = (x^2 - 1)(x - 2) \text{ on } [-1, 2]$$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e, on R.

Let us find the values at extremums:

$$\Rightarrow f(-1) = ((-1)^2 - 1)(-1 - 2)$$

$$\Rightarrow f(-1) = (1 - 1)(-3)$$

$$\Rightarrow f(-1) = (0)(-3)$$

$$\Rightarrow f(-1) = 0$$

$$\Rightarrow f(2) = (2^2 - 1)(2 - 2)$$

$$\Rightarrow f(2) = (4 - 1)(0)$$

$$\Rightarrow f(2) = 0$$

$\therefore f(-1) = f(2)$, Rolle's theorem applicable for function 'f' on $[-1, 2]$.

Let's find the derivative of f(x):

$$\Rightarrow f'(x) = \frac{d((x^2 - 1)(x - 2))}{dx}$$

Differentiating using UV rule,

$$\Rightarrow f'(x) = (x - 2) \times \frac{d(x^2 - 1)}{dx} + (x^2 - 1) \frac{d(x - 2)}{dx}$$

$$\Rightarrow f'(x) = ((x - 2) \times 2x) + ((x^2 - 1) \times 1)$$

$$\Rightarrow f'(x) = 2x^2 - 4x + x^2 - 1$$

$$\Rightarrow f'(x) = 2x^2 - 4x - 1$$

We have $f'(c) = 0$ $c \in (-1, 2)$, from the definition given above.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c^2 - 4c - 1 = 0$$

$$\Rightarrow c = \frac{4 \pm \sqrt{(-4)^2 - (4 \times 2 \times -1)}}{2 \times 2}$$

$$\Rightarrow c = \frac{4 \pm \sqrt{16+8}}{4}$$

$$\Rightarrow c = \frac{4 \pm \sqrt{24}}{4}$$

$$\Rightarrow c = \frac{4+2\sqrt{6}}{4} \text{ or } c = \frac{4-2\sqrt{6}}{4}$$

$$\Rightarrow c = 1 + \frac{\sqrt{6}}{2} \text{ or } c = 1 - \frac{\sqrt{6}}{2}$$

$$\Rightarrow c = 1 - \frac{\sqrt{6}}{2} \in (-1, 2)$$

$\therefore$ Rolle's theorem is verified.

2 F. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = x(x-4)^2 \text{ on } [0, 4]$$

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = x(x-4)^2 \text{ on } [0,4]$$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on R.

Let us find the values at extremums:

$$\Rightarrow f(0) = 0(0-4)^2$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(4) = 4(4-4)^2$$

$$\Rightarrow f(4) = 4(0)^2$$

$$\Rightarrow f(4) = 0$$

$\therefore f(0) = f(4)$, Rolle's theorem applicable for function 'f' on [0,4].

Let's find the derivative of f(x):

$$\Rightarrow f'(x) = \frac{d(x(x-4)^2)}{dx}$$

Differentiating using UV rule,

$$\Rightarrow f'(x) = (x-4)^2 \times \frac{d(x)}{dx} + x \frac{d((x-4)^2)}{dx}$$

$$\Rightarrow f'(x) = ((x-4)^2 \times 1) + (x \times 2 \times (x-4))$$

$$\Rightarrow f'(x) = (x-4)^2 + 2(x^2 - 4x)$$

$$\Rightarrow f'(x) = x^2 - 8x + 16 + 2x^2 - 8x$$

$$\Rightarrow f'(x) = 3x^2 - 16x + 16$$

We have $f'(c) = 0 \quad c \in (0,4)$, from the definition given above.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 3c^2 - 16c + 16 = 0$$

$$\Rightarrow c = \frac{16 \pm \sqrt{(-16)^2 - (4 \times 3 \times 16)}}{2 \times 3}$$

$$\Rightarrow c = \frac{16 \pm \sqrt{256 - 192}}{6}$$

$$\Rightarrow c = \frac{16 \pm \sqrt{64}}{6}$$

$$\Rightarrow c = \frac{8}{6} \text{ or } c = \frac{24}{6}$$

$$\Rightarrow c = \frac{8}{6} \in (0,4)$$

$\therefore$ Rolle's theorem is verified.

2 G. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = x(x - 2)^2 \text{ on } [0, 2]$$

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a,b]$.

b) The function 'f' needs differentiable on the open interval (a,b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = x(x - 2)^2 \text{ on } [0,2]$$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e, on $\mathbb{R}$.

Let us find the values at extremums:

$$\Rightarrow f(0) = 0(0 - 2)^2$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(2) = 2(2 - 2)^2$$

$$\Rightarrow f(2) = 2(0)^2$$

$$\Rightarrow f(2) = 0$$

$\therefore f(0) = f(2)$, Rolle's theorem applicable for function 'f' on $[0,2]$.

Let's find the derivative of $f(x)$:

$$\Rightarrow f'(x) = \frac{d(x(x-2)^2)}{dx}$$

Differentiating using UV rule,

$$\Rightarrow f'(x) = (x - 2)^2 \times \frac{d(x)}{dx} + x \frac{d((x-2)^2)}{dx}$$

$$\Rightarrow f'(x) = ((x-2)^2 \times 1) + (x \times 2 \times (x-2))$$

$$\Rightarrow f'(x) = (x-2)^2 + 2(x^2 - 2x)$$

$$\Rightarrow f'(x) = x^2 - 4x + 4 + 2x^2 - 4x$$

$$\Rightarrow f'(x) = 3x^2 - 8x + 4$$

We have $f'(c) = 0$ $c \in (0,1)$, from the definition given above.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 3c^2 - 8c + 4 = 0$$

$$\Rightarrow c = \frac{8 \pm \sqrt{(-8)^2 - (4 \times 3 \times 4)}}{2 \times 3}$$

$$\Rightarrow c = \frac{8 \pm \sqrt{64 - 48}}{6}$$

$$\Rightarrow c = \frac{8 \pm \sqrt{16}}{6}$$

$$\Rightarrow c = \frac{12}{6} \text{ OR } c = \frac{6}{6}$$

$$\Rightarrow c = 1 \in (0,2)$$

$\therefore$ Rolle's theorem is verified.

2 H. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = x^2 + 5x + 6 \text{ on } [-3, -2]$$

Answer

First let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a,b]$.

b) The function 'f' needs differentiable on the open interval (a,b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = x^2 + 5x + 6 \text{ on } [-3, -2]$$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on $\mathbb{R}$.

Let us find the values at extremums:

$$\Rightarrow f(-3) = (-3)^2 + 5(-3) + 6$$

$$\Rightarrow f(-3) = 9 - 15 + 6$$

$$\Rightarrow f(-3) = 0$$

$$\Rightarrow f(-2) = (-2)^2 + 5(-2) + 6$$

$$\Rightarrow f(-2) = 4 - 10 + 6$$

$$\Rightarrow f(-2) = 0$$

$\therefore f(-3) = f(-2)$, Rolle's theorem applicable for function 'f' on $[-3, -2]$.

Let's find the derivative of $f(x)$:

$$\Rightarrow f'(x) = \frac{d(x^2 + 5x + 6)}{dx}$$

$$\Rightarrow f'(x) = \frac{d(x^2)}{dx} + \frac{d(5x)}{dx} + \frac{d(6)}{dx}$$

$$\Rightarrow f'(x) = 2x + 5 + 0$$

$$\Rightarrow f'(x) = 2x + 5$$

We have $f'(c) = 0 \Leftrightarrow c \in (-3, -2)$, from the definition given above.

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c + 5 = 0$$

$$\Rightarrow 2c = -5$$

$$\Rightarrow c = -\frac{5}{2}$$

$$\Rightarrow C = -2.5 \in (-3, -2)$$

$\therefore$ Rolle's theorem is verified.

3 A. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \cos 2 \left(x - \frac{\pi}{4} \right) \text{ on } [0, \frac{\pi}{2}]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is :

$$\Rightarrow f(x) = \cos 2 \left(x - \frac{\pi}{4} \right) \text{ on } \left[0, \frac{\pi}{2} \right]$$

We know that cosine function is continuous and differentiable on $\mathbb{R}$.

Let's find the values of the function at an extremum,

$$\Rightarrow f(0) = \cos 2 \left(0 - \frac{\pi}{4} \right)$$

$$\Rightarrow f(0) = \cos 2 \left(-\frac{\pi}{4} \right)$$

$$\Rightarrow f(0) = \cos \left(-\frac{\pi}{2} \right)$$

We know that $\cos(-x) = \cos x$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos 2 \left(\frac{\pi}{2} - \frac{\pi}{4} \right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos 2 \left(\frac{\pi}{4} \right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos \left(\frac{\pi}{2} \right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$$

We got $f(0) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d\left(\cos 2\left(x - \frac{\pi}{4}\right)\right)}{dx}$$

$$\Rightarrow f'(x) = -\sin\left(2\left(x - \frac{\pi}{4}\right)\right) \frac{d\left(2\left(x - \frac{\pi}{4}\right)\right)}{dx}$$

$$\Rightarrow f'(x) = -2 \sin 2\left(x - \frac{\pi}{4}\right)$$

We have $f'(c) = 0$,

$$\Rightarrow -2 \sin 2\left(c - \frac{\pi}{4}\right) = 0$$

$$\Rightarrow c - \frac{\pi}{4} = 0$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

$\therefore$ Rolle's theorem is verified.

3 B. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$f(x) = \sin 2x$ on $[0, \pi/2]$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = \sin 2x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that sine function is continuous and differentiable on $\mathbb{R}$.

Let's find the values of function at extremum,

$$\Rightarrow f(0) = \sin 2(0)$$

$$\Rightarrow f(0) = \sin 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin 2\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin(\pi)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$$

We got $f(0) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(\sin 2x)}{dx}$$

$$\Rightarrow f'(x) = \cos 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = 2\cos 2x$$

We have $f'(c) = 0$,

$$\Rightarrow 2\cos 2c = 0$$

$$\Rightarrow 2c = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

$\therefore$ Rolle's theorem is verified.

3 C. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$f(x) = \cos 2x$ on $[-\pi/4, \pi/4]$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

- a) The function 'f' needs to be continuous in the closed interval $[a, b]$.
- b) The function 'f' needs differentiable on the open interval (a, b) .
- c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow \cos 2x \text{ on } \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$$

We know that cosine function is continuous and differentiable on $\mathbb{R}$.

Let's find the values of the function at an extremum,

$$\Rightarrow f\left(-\frac{\pi}{4}\right) = \cos 2\left(-\frac{\pi}{4}\right)$$

$$\Rightarrow f(0) = \cos\left(-\frac{\pi}{2}\right)$$

We know that $\cos(-x) = \cos x$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f\left(\frac{\pi}{4}\right) = \cos 2\left(\frac{\pi}{4}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 0$$

We got $f\left(-\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right)$, so there exist a $c \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(\cos 2x)}{dx}$$

$$\Rightarrow f'(x) = -\sin 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = -2\sin 2x$$

We have $f'(c) = 0$,

$$\Rightarrow -2\sin 2c = 0$$

$$\Rightarrow 2c = 0$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

$\therefore$ Rolle's theorem is verified.

3 D. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = e^x \sin x \text{ on } [0, \pi]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = e^x \sin x \text{ on } [0, \pi]$$

We know that exponential and sine functions are continuous and differentiable on $\mathbb{R}$.

Let's find the values of the function at an extremum,

$$\Rightarrow f(0) = e^0 \sin(0)$$

$$\Rightarrow f(0) = 1 \times 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(\pi) = e^\pi \sin(\pi)$$

$$\Rightarrow f(\pi) = e^\pi \times 0$$

$$\Rightarrow f(\pi) = 0$$

We got $f(0) = f(\pi)$, so there exist a $c \in (0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(e^x \sin x)}{dx}$$

$$\Rightarrow f'(x) = \sin x \frac{d(e^x)}{dx} + e^x \frac{d(\sin x)}{dx}$$

$$\Rightarrow f'(x) = e^x (\sin x + \cos x)$$

We have $f'(c) = 0$,

$$\Rightarrow e^c(\sin c + \cos c) = 0$$

$$\Rightarrow \sin c + \cos c = 0$$

$$\Rightarrow \frac{1}{\sqrt{2}}\sin c + \frac{1}{\sqrt{2}}\cos c = 0$$

$$\Rightarrow \sin\left(\frac{\pi}{4}\right)\sin c + \cos\left(\frac{\pi}{4}\right)\cos c = 0$$

$$\Rightarrow \cos\left(c - \frac{\pi}{4}\right) = 0$$

$$\Rightarrow c - \frac{\pi}{4} = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{3\pi}{4} \in (0, \pi)$$

$\therefore$ Rolle's theorem is verified.

3 E. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = e^x \cos x \text{ on } [-\pi/2, \pi/2]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = e^x \cos x \text{ on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

We know that exponential and cosine functions are continuous and differentiable on $\mathbb{R}$.

Let's find the values of the function at an extremum,

$$\Rightarrow f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}} \cos\left(-\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(-\frac{\pi}{2}\right) = e^{-\frac{\pi}{2}} \times 0$$

$$\Rightarrow f\left(-\frac{\pi}{2}\right) = 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = e^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f(\pi) = e^{\frac{\pi}{2}} \times 0$$

$$\Rightarrow f(\pi) = 0$$

We got $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right)$, so there exist a $c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(e^x \cos x)}{dx}$$

$$\Rightarrow f'(x) = \cos x \frac{d(e^x)}{dx} + e^x \frac{d(\cos x)}{dx}$$

$$\Rightarrow f'(x) = e^x(-\sin x + \cos x)$$

We have $f'(c) = 0$,

$$\Rightarrow e^c(-\sin c + \cos c) = 0$$

$$\Rightarrow -\sin c + \cos c = 0$$

$$\Rightarrow \frac{-1}{\sqrt{2}} \sin c + \frac{1}{\sqrt{2}} \cos c = 0$$

$$\Rightarrow -\sin\left(\frac{\pi}{4}\right) \sin c + \cos\left(\frac{\pi}{4}\right) \cos c = 0$$

$$\Rightarrow \cos\left(c + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow c + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$\therefore$ Rolle's theorem is verified.

3 F. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$f(x) = \cos 2x$ on $[0, \pi]$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = \cos 2x \text{ on } [0, \pi]$$

We know that cosine function is continuous and differentiable on $\mathbb{R}$.

Let's find the values of function at extremum,

$$\Rightarrow f(0) = \cos 2(0)$$

$$\Rightarrow f(0) = \cos(0)$$

$$\Rightarrow f(0) = 1$$

$$\Rightarrow f(\pi) = \cos 2(\pi)$$

$$\Rightarrow f(\pi) = \cos(2\pi)$$

$$\Rightarrow f(\pi) = 1$$

We got $f(0) = f(\pi)$, so there exist a $c \in (0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(\cos 2x)}{dx}$$

$$\Rightarrow f'(x) = -\sin 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = -2\sin 2x$$

We have $f'(c) = 0$,

$$\Rightarrow -2\sin 2c = 0$$

$$\Rightarrow 2c = 0$$

$$\Rightarrow c = \frac{\pi}{4} \in (0, \pi)$$

$\therefore$ Rolle's theorem is verified.

3 G. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \frac{\sin x}{e^x} \text{ on } 0 \leq x \leq \pi$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = \frac{\sin x}{e^x} \text{ on } [0, \pi]$$

This can be written as

$$\Rightarrow f(x) = e^{-x} \sin x \text{ on } [0, \pi]$$

We know that exponential and sine functions are continuous and differentiable on $\mathbb{R}$.

Let's find the values of the function at an extremum,

$$\Rightarrow f(0) = e^{-0} \sin(0)$$

$$\Rightarrow f(0) = 1 \times 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(\pi) = e^{-\pi} \sin(\pi)$$

$$\Rightarrow f(\pi) = e^{-\pi} \times 0$$

$$\Rightarrow f(\pi) = 0$$

We got $f(0) = f(\pi)$, so there exist a $c \in (0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(e^{-x} \sin x)}{dx}$$

$$\Rightarrow f'(x) = \sin x \frac{d(e^{-x})}{dx} + e^{-x} \frac{d(\sin x)}{dx}$$

$$\Rightarrow f'(x) = \sin x(-e^{-x}) + e^{-x}(\cos x)$$

$$\Rightarrow f'(x) = e^{-x}(-\sin x + \cos x)$$

We have $f'(c) = 0$,

$$\Rightarrow e^{-c}(-\sin c + \cos c) = 0$$

$$\Rightarrow -\sin c + \cos c = 0$$

$$\Rightarrow -\frac{1}{\sqrt{2}}\sin c + \frac{1}{\sqrt{2}}\cos c = 0$$

$$\Rightarrow -\sin\left(\frac{\pi}{4}\right)\sin c + \cos\left(\frac{\pi}{4}\right)\cos c = 0$$

$$\Rightarrow \cos\left(c + \frac{\pi}{4}\right) = 0$$

$$\Rightarrow c + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{\pi}{4} \in (0, \pi)$$

$\therefore$ Rolle's theorem is verified.

3 H. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \sin 3x \text{ on } [0, \pi]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = \sin 3x \text{ on } [0, \pi]$$

We know that sine function is continuous and differentiable on $\mathbb{R}$.

Let's find the values of function at extremum,

$$\Rightarrow f(0) = \sin 3(0)$$

$$\Rightarrow f(0) = \sin 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(\pi) = \sin 3(\pi)$$

$$\Rightarrow f(\pi) = \sin(3\pi)$$

$$\Rightarrow f(\pi) = 0$$

We got $f(0) = f(\pi)$, so there exist a $c \in (0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(\sin 3x)}{dx}$$

$$\Rightarrow f'(x) = \cos 3x \frac{d(3x)}{dx}$$

$$\Rightarrow f'(x) = 3\cos 3x$$

We have $f'(c) = 0$,

$$\Rightarrow 3\cos 3c = 0$$

$$\Rightarrow 3c = \frac{\pi}{2}$$

$$\Rightarrow c = \frac{\pi}{6} \in (0, \pi)$$

$\therefore$ Rolle's theorem is verified.

3 I. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = e^{1-x^2} \text{ on } [-1, 1]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = e^{1-x^2} \text{ on } [-1, 1]$$

We know that exponential function is continuous and differentiable over $\mathbb{R}$.

Let's find the value of function f at extremums,

$$\Rightarrow f(-1) = e^{1-(-1)^2}$$

$$\Rightarrow f(-1) = e^{1-1}$$

$$\Rightarrow f(-1) = e^0$$

$$\Rightarrow f(-1) = 1$$

$$\Rightarrow f(1) = e^{1-1^2}$$

$$\Rightarrow f(1) = e^{1-1}$$

$$\Rightarrow f(1) = e^0$$

$$\Rightarrow f(1) = 1$$

We got $f(-1) = f(1)$ so, there exists a $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's find the derivative of the function f :

$$\Rightarrow f'(x) = \frac{d(e^{1-x^2})}{dx}$$

$$\Rightarrow f'(x) = e^{1-x^2} \frac{d(1-x^2)}{dx}$$

$$\Rightarrow f'(x) = e^{1-x^2}(-2x)$$

We have $f'(c) = 0$

$$\Rightarrow e^{1-c^2}(-2c) = 0$$

$$\Rightarrow 2c = 0$$

$$\Rightarrow c = 0 \in [-1, 1]$$

$\therefore$ Rolle's theorem is verified.

3 J. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \log(x^2 + 2) - \log 3 \text{ on } [-1, 1]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = \log(x^2 + 2) - \log 3 \text{ on } [-1, 1]$$

We know that logarithmic function is continuous and differentiable in its own domain.

We check the values of the function at the extremum,

$$\Rightarrow f(-1) = \log((-1)^2 + 2) - \log 3$$

$$\Rightarrow f(-1) = \log(1 + 2) - \log 3$$

$$\Rightarrow f(-1) = \log 3 - \log 3$$

$$\Rightarrow f(-1) = 0$$

$$\Rightarrow f(1) = \log(1^2 + 2) - \log 3$$

$$\Rightarrow f(1) = \log(1 + 2) - \log 3$$

$$\Rightarrow f(1) = \log 3 - \log 3$$

$$\Rightarrow f(1) = 0$$

We have got $f(-1) = f(1)$. So, there exists a c such that $c \in (-1, 1)$ such that $f'(c) = 0$.

Let's find the derivative of the function f ,

$$\Rightarrow f'(x) = \frac{d(\log(x^2 + 2) - \log 3)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{x^2 + 2} \frac{d(x^2 + 2)}{dx} - 0$$

$$\Rightarrow f'(x) = \frac{2x}{x^2 + 2}$$

We have $f'(c) = 0$

$$\Rightarrow \frac{2c}{c^2 + 2} = 0$$

$$\Rightarrow 2c = 0$$

$$\Rightarrow c = 0 \in (-1, 1)$$

$\therefore$ Rolle's theorem is verified.

3 K. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \sin x + \cos x \text{ on } [0, \pi/2]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

$$c) f(a) = f(b)$$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = \sin x + \cos x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that sine and cosine functions are continuous and differentiable on R.

Let's the value of function f at extremums:

$$\Rightarrow f(0) = \sin(0) + \cos(0)$$

$$\Rightarrow f(0) = 0 + 1$$

$$\Rightarrow f(0) = 1$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 + 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1$$

We have got $f(0) = f\left(\frac{\pi}{2}\right)$. So, there exists a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Let's find the derivative of the function 'f'.

$$\Rightarrow f'(x) = \frac{d(\sin x + \cos x)}{dx}$$

$$\Rightarrow f'(x) = \cos x - \sin x$$

$$\text{We have } f'(c) = 0$$

$$\Rightarrow \cos c - \sin c = 0$$

$$\Rightarrow \frac{1}{\sqrt{2}} \cos c - \frac{1}{\sqrt{2}} \sin c = 0$$

$$\Rightarrow \sin\left(\frac{\pi}{4}\right) \cos c - \cos\left(\frac{\pi}{4}\right) \sin c = 0$$

$$\Rightarrow \sin\left(\frac{\pi}{4} - c\right) = 0$$

$$\Rightarrow \frac{\pi}{4} - c = 0$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

$\therefore$ Rolle's theorem is verified.

3 L. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = 2 \sin x + \sin 2x \text{ on } [0, \pi]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = 2\sin x + \sin 2x \text{ on } [0, \pi]$$

We know that sine function continuous and differentiable over $\mathbb{R}$.

Let's check the values of function f at the extremums

$$\Rightarrow f(0) = 2\sin(0) + \sin 2(0)$$

$$\Rightarrow f(0) = 2(0) + 0$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(\pi) = 2\sin(\pi) + \sin 2(\pi)$$

$$\Rightarrow f(\pi) = 2(0) + 0$$

$$\Rightarrow f(\pi) = 0$$

We have got $f(0) = f(\pi)$. So, there exists a $c \in (0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of function 'f'.

$$\Rightarrow f'(x) = \frac{d(2\sin x + \sin 2x)}{dx}$$

$$\Rightarrow f'(x) = 2\cos x + \cos 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = 2\cos x + 2\cos 2x$$

$$\Rightarrow f'(x) = 2\cos x + 2(2\cos^2 x - 1)$$

$$\Rightarrow f'(x) = 4\cos^2 x + 2\cos x - 2$$

We have $f'(c) = 0$,

$$\Rightarrow 4\cos^2 c + 2\cos c - 2 = 0$$

$$\Rightarrow 2\cos^2 c + \cos c - 1 = 0$$

$$\Rightarrow 2\cos^2 c + 2\cos c - \cos c - 1 = 0$$

$$\Rightarrow 2\cos c(\cos c + 1) - 1(\cos c + 1) = 0$$

$$\Rightarrow (2\cos c - 1)(\cos c + 1) = 0$$

$$\Rightarrow \cos c = \frac{1}{2} \text{ or } \cos c = -1$$

$$\Rightarrow c = \frac{\pi}{3} \in (0, \pi)$$

$\therefore$ Rolle's theorem is verified.

3 M. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \frac{x}{2} - \sin \frac{\pi x}{6} \text{ on } [-1, 0]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = \frac{x}{2} - \sin\left(\frac{\pi x}{6}\right) \text{ on } [-1, 0]$$

We know that sine function is continuous and differentiable over R.

Let's check the values of 'f' at an extremum

$$\Rightarrow f(-1) = \frac{-1}{2} - \sin\left(\frac{\pi(-1)}{6}\right)$$

$$\Rightarrow f(-1) = -\frac{1}{2} - \sin\left(\frac{-\pi}{6}\right)$$

$$\Rightarrow f(-1) = -\frac{1}{2} - \left(-\frac{1}{2}\right)$$

$$\Rightarrow f(-1) = 0$$

$$\Rightarrow f(0) = \frac{0}{2} - \sin\left(\frac{\pi(0)}{6}\right)$$

$$\Rightarrow f(0) = 0 - \sin(0)$$

$$\Rightarrow f(0) = 0 - 0$$

$$\Rightarrow f(0) = 0$$

We have got $f(-1) = f(0)$. So, there exists a $c \in (-1, 0)$ such that $f'(c) = 0$.

Let's find the derivative of the function 'f'

$$\Rightarrow f'(x) = \frac{d\left(\frac{x}{2} - \sin\left(\frac{\pi x}{6}\right)\right)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2} - \cos\left(\frac{\pi x}{6}\right) \frac{d\left(\frac{\pi x}{6}\right)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2} - \frac{\pi}{6} \cos\left(\frac{\pi x}{6}\right)$$

We have $f'(c) = 0$

$$\Rightarrow \frac{1}{2} - \frac{\pi}{6} \cos\left(\frac{\pi c}{6}\right) = 0$$

$$\Rightarrow \frac{\pi}{6} \cos\left(\frac{\pi c}{6}\right) = \frac{1}{2}$$

$$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = \frac{1}{2} \times \frac{6}{\pi}$$

$$\Rightarrow \cos\left(\frac{\pi c}{6}\right) = \frac{3}{\pi}$$

$$\Rightarrow \frac{\pi c}{6} = \cos^{-1}\left(\frac{3}{\pi}\right)$$

$$\Rightarrow c = \frac{6}{\pi} \cos^{-1}\left(\frac{3}{\pi}\right)$$

Cosine is positive between $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, for our convenience we take the interval to be $-\frac{\pi}{2} \leq \theta \leq 0$, since the values of the cosine repeats.

We know that $\frac{3}{\pi}$ value is nearly equal to 1. So, the value of the c nearly equal to 0.

So, we can clearly say that $c \in (-1, 0)$.

$\therefore$ Rolle's theorem is verified.

3 N. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \frac{6x}{\pi} - 4 \sin^2 x \text{ on } \left[0, \frac{\pi}{2}\right]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = \frac{6x}{\pi} - 4 \sin^2 x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that sine function is continuous and differentiable over R.

Let's check the values of function 'f' at the extremums,

$$\Rightarrow f(0) = \frac{6(0)}{\pi} - 4 \sin^2(0)$$

$$\Rightarrow f(0) = 0 - 4(0)$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \frac{6\left(\frac{\pi}{2}\right)}{\pi} - 4 \sin^2\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi}{\pi} - 4\left(\frac{1}{2}\right)^2$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 - 4\left(\frac{1}{4}\right)$$

$$\Rightarrow f\left(\frac{\pi}{6}\right) = 1 - 1$$

$$\Rightarrow f\left(\frac{\pi}{6}\right) = 0.$$

We got $f(0) = f\left(\frac{\pi}{6}\right)$. So, there exists a $c \in \left(0, \frac{\pi}{6}\right)$ such that $f'(c) = 0$.

Let's find the derivative of function 'f.'

$$\Rightarrow f'(x) = \frac{d\left(\frac{6x}{\pi} - 4\sin^2 x\right)}{dx}$$

$$\Rightarrow f'(x) = \frac{6}{\pi} - 4 \times 2\sin x \times \frac{d(\sin x)}{dx}$$

$$\Rightarrow f'(x) = \frac{6}{\pi} - 8\sin x(\cos x)$$

$$\Rightarrow f'(x) = \frac{6}{\pi} - 4(2\sin x \cos x)$$

$$\Rightarrow f'(x) = \frac{6}{\pi} - 4\sin 2x$$

We have $f'(c) = 0$

$$\Rightarrow \frac{6}{\pi} - 4\sin 2c = 0$$

$$\Rightarrow 4\sin 2c = \frac{6}{\pi}$$

$$\Rightarrow \sin 2c = \frac{6}{4\pi}$$

We know $\frac{6}{4\pi} < \frac{1}{2}$

$$\Rightarrow \sin 2c < \frac{1}{2}$$

$$\Rightarrow 2c < \sin^{-1}\left(\frac{1}{2}\right)$$

$$\Rightarrow 2c < \frac{\pi}{6}$$

$$\Rightarrow c < \frac{\pi}{12} \in \left(0, \frac{\pi}{6}\right)$$

$\therefore$ Rolle's theorem is verified.

3 O. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = 4^{\sin x} \text{ on } [0, \pi]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

- The function 'f' needs to be continuous in the closed interval $[a, b]$.
- The function 'f' needs differentiable on the open interval (a, b) .
- $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = 4^{\sin x} \text{ on } [0, \pi]$$

We that sine function is continuous and differentiable over $\mathbb{R}$.

Let's check the values of function 'f' at extremums

$$\Rightarrow f(0) = 4^{\sin(0)}$$

$$\Rightarrow f(0) = 4^0$$

$$\Rightarrow f(0) = 1$$

$$\Rightarrow f(\pi) = 4^{\sin \pi}$$

$$\Rightarrow f(\pi) = 4^0$$

$$\Rightarrow f(\pi) = 1$$

We got $f(0) = f(\pi)$. So, there exists a $c \in (0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of 'f'

$$\Rightarrow f'(x) = \frac{d(4^{\sin x})}{dx}$$

$$\Rightarrow f'(x) = 4^{\sin x} \log 4 \frac{d(\sin x)}{dx}$$

$$\Rightarrow f'(x) = 4^{\sin x} \log 4 \cos x$$

We have $f'(c) = 0$

$$\Rightarrow 4^{\sin c} \log 4 \cos c = 0$$

$$\Rightarrow \cos c = 0$$

$$\Rightarrow c = \frac{\pi}{2} \in (0, \pi)$$

$\therefore$ Rolle's theorem is verified.

3 P. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = x^2 - 5x + 4 \text{ on } [1, 4]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = x^2 - 5x + 4 \text{ on } [1, 4]$$

Since, given function f is a polynomial it is continuous and differentiable everywhere i.e., on $\mathbb{R}$.

Let us find the values at extremums:

$$\Rightarrow f(1) = 1^2 - 5(1) + 4$$

$$\Rightarrow f(1) = 1 - 5 + 4$$

$$\Rightarrow f(1) = 0$$

$$\Rightarrow f(4) = 4^2 - 5(4) + 4$$

$$\Rightarrow f(4) = 16 - 20 + 4$$

$$\Rightarrow f(4) = 0$$

$\therefore$ We got $f(1) = f(4)$. So, there exists a $c \in (1,4)$ such that $f'(c) = 0$.

Let's find the derivative of $f(x)$:

$$\Rightarrow f'(x) = \frac{d(x^2 - 5x + 4)}{dx}$$

$$\Rightarrow f'(x) = \frac{d(x^2)}{dx} - \frac{d(5x)}{dx} + \frac{d(4)}{dx}$$

$$\Rightarrow f'(x) = 2x - 5 + 0$$

$$\Rightarrow f'(x) = 2x - 5$$

We have $f'(c) = 0$

$$\Rightarrow f'(c) = 0$$

$$\Rightarrow 2c - 5 = 0$$

$$\Rightarrow 2c = 5$$

$$\Rightarrow c = \frac{5}{2}$$

$$\Rightarrow C = 2.5 \in (1,4)$$

$\therefore$ Rolle's theorem is verified.

3 Q. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$$f(x) = \sin^4 x + \cos^4 x \text{ on } \left[0, \frac{\pi}{2}\right]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a,b]$.

b) The function 'f' needs differentiable on the open interval (a,b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = \sin^4 x + \cos^4 x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that sine and cosine functions are continuous and differentiable functions over $\mathbb{R}$.

Let's find the value of function 'f' at extremums

$$\Rightarrow f(0) = \sin^4(0) + \cos^4(0)$$

$$\Rightarrow f(0) = (0)^4 + (1)^4$$

$$\Rightarrow f(0) = 0 + 1$$

$$\Rightarrow f(0) = 1$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1^4 + 0^4$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1 + 0$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = 1$$

We got $f(0) = f\left(\frac{\pi}{2}\right)$. So, there exists a $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Let's find the derivative of the function 'f'.

$$\Rightarrow f'(x) = \frac{d(\sin^4 x + \cos^4 x)}{dx}$$

$$\Rightarrow f'(x) = 4\sin^3 x \frac{d(\sin x)}{dx} + 4\cos^3 x \frac{d(\cos x)}{dx}$$

$$\Rightarrow f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$$

$$\Rightarrow f'(x) = 4\sin x \cos x (\sin^2 x - \cos^2 x)$$

$$\Rightarrow f'(x) = 2(2\sin x \cos x)(-\cos 2x)$$

$$\Rightarrow f'(x) = -2(\sin 2x)(\cos 2x)$$

$$\Rightarrow f'(x) = -\sin 4x$$

We have $f'(c) = 0$

$$\Rightarrow -\sin 4c = 0$$

$$\Rightarrow \sin 4c = 0$$

$$\Rightarrow 4c = 0 \text{ or } \pi$$

$$\Rightarrow c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

$\therefore$ Rolle's theorem is verified.

3 R. Question

Verify Rolle's theorem for each of the following functions on the indicated intervals :

$f(x) = \sin x - \sin 2x$ on $[0, \pi]$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = \sin x - \sin 2x \text{ on } [0, \pi]$$

We know that sine function is continuous and differentiable over $\mathbb{R}$.

Let's check the values of the function 'f' at the extremums.

$$\Rightarrow f(0) = \sin(0) - \sin^2(0)$$

$$\Rightarrow f(0) = 0 - \sin(0)$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(\pi) = \sin(\pi) - \sin^2(\pi)$$

$$\Rightarrow f(\pi) = 0 - \sin^2(\pi)$$

$$\Rightarrow f(\pi) = 0$$

We got $f(0) = f(\pi)$. So, there exists a $c \in (0, \pi)$ such that $f'(c) = 0$.

Let's find the derivative of the function 'f'

$$\Rightarrow f'(x) = \frac{d(\sin x - \sin^2 x)}{dx}$$

$$\Rightarrow f'(x) = \cos x - \cos 2x \frac{d(2x)}{dx}$$

$$\Rightarrow f'(x) = \cos x - 2\cos 2x$$

$$\Rightarrow f'(x) = \cos x - 2(2\cos^2 x - 1)$$

$$\Rightarrow f'(x) = \cos x - 4\cos^2 x + 2$$

We have $f'(c) = 0$

$$\Rightarrow \cos c - 4\cos^2 c + 2 = 0$$

$$\Rightarrow \cos c = \frac{-1 \pm \sqrt{(1)^2 - (4 \times -4 \times 2)}}{2 \times -4}$$

$$\Rightarrow \cos c = \frac{-1 \pm \sqrt{1 + 32}}{-8}$$

$$\Rightarrow c = \cos^{-1}\left(\frac{-1 \pm \sqrt{33}}{-8}\right)$$

We can see that $c \in (0, \pi)$

$\therefore$ Rolle's theorem is verified.

4. Question

Using Rolle's theorem, find points on the curve $y = 16 - x^2$, $x \in [-1, 1]$, where the tangent is parallel to the x-axis.

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow y = 16 - x^2, x \in [-1, 1]$$

We know that polynomial function is continuous and differentiable over $\mathbb{R}$.

Let's check the values of 'y' at extremums

$$\Rightarrow y(-1) = 16 - (-1)^2$$

$$\Rightarrow y(-1) = 16 - 1$$

$$\Rightarrow y(-1) = 15$$

$$\Rightarrow y(1) = 16 - (1)^2$$

$$\Rightarrow y(1) = 16 - 1$$

$$\Rightarrow y(1) = 15$$

We got $y(-1) = y(1)$. So, there exists a $c \in (-1, 1)$ such that $f'(c) = 0$.

We know that for a curve g , the value of the slope of the tangent at a point r is given by $g'(r)$.

Let's find the derivative of curve y

$$\Rightarrow y' = \frac{d(16-x^2)}{dx}$$

$$\Rightarrow y' = -2x$$

We have $y'(c) = 0$

$$\Rightarrow -2c = 0$$

$$\Rightarrow c = 0 \in (-1, 1)$$

Value of y at $x = 1$ is

$$\Rightarrow y = 16 - 0^2$$

$$\Rightarrow y = 16$$

$\therefore$ The point at which the curve y has a tangent parallel to x -axis (since the slope of x -axis is 0) is $(0, 16)$.

5 A. Question

At what points on the following curves, is the tangent parallel to the x -axis?

$$y = x^2 \text{ on } [-2, 2]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function ' f ':

a) The function ' f ' needs to be continuous in the closed interval $[a, b]$.

b) The function ' f ' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow y = x^2 \text{ on } [-2, 2]$$

We know that polynomials are continuous and differentiable over $\mathbb{R}$.

Let's check the values of y at the extremums

$$\Rightarrow y(-2) = (-2)^2$$

$$\Rightarrow y(-2) = 4$$

$$\Rightarrow y(2) = (2)^2 \Rightarrow y(2) = 4$$

We got $y(-2) = y(2)$. So, there exists a c such that $f'(c) = 0$.

For a curve g to have a tangent parallel to x - axis at point r , the criteria to be satisfied is $g'(r) = 0$.

$$\Rightarrow y'(x) = 0$$

$$\Rightarrow \frac{d(x^2)}{dx} = 0$$

$$\Rightarrow 2x = 0$$

$$\Rightarrow x = 0$$

The value of y is

$$\Rightarrow y = (0)^2$$

$$\Rightarrow y = 0$$

The point at which the curve has tangent parallel to x - axis is $(0,0)$.

5 B. Question

At what points on the following curves, is the tangent parallel to the x -axis?

$$y = e^{1-x^2} \text{ on } [-1, 1]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function ' f ':

a) The function ' f ' needs to be continuous in the closed interval $[a,b]$.

b) The function ' f ' needs differentiable on the open interval (a,b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow y = e^{1-x^2} \text{ on } [-1, 1]$$

We know that exponential functions are continuous and differentiable over $\mathbb{R}$.

Let's check the values of y at the extremums

$$\Rightarrow y(-1) = e^{1-(-1)^2}$$

$$\Rightarrow y(-1) = e^{1-1}$$

$$\Rightarrow y(-1) = e^0 \Rightarrow y(-1) = 1$$

$$\Rightarrow y(1) = e^{1-1^2}$$

$$\Rightarrow y(1) = e^{1-1}$$

$$\Rightarrow y(1) = e^0$$

$$\Rightarrow y(1) = 1$$

We got $y(-1) = y(1)$. So, there exists a c such that $f'(c) = 0$.

For a curve g to have a tangent parallel to the x - axis at point r , the criteria to be satisfied is $g'(r) = 0$.

$$\Rightarrow y'(x) = 0$$

$$\Rightarrow \frac{d(e^{1-x^2})}{dx} = 0$$

$$\Rightarrow e^{1-x^2} \frac{d(1-x^2)}{dx} = 0$$

$$\Rightarrow e^{1-x^2} (-2x) = 0$$

$$\Rightarrow 2x = 0$$

$$\Rightarrow x = 0$$

The value of y is

$$\Rightarrow y = e^{1-0^2}$$

$$\Rightarrow y = e^{1-0}$$

$$\Rightarrow y = e^1$$

$$\Rightarrow y = e$$

The point at which the curve has a tangent parallel to the x - axis is (0,e).

5 C. Question

At what points on the following curves, is the tangent parallel to the x-axis?

$$y = 12(x + 1)(x - 2) \text{ on } [-1, 2]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval [a,b].

b) The function 'f' needs differentiable on the open interval (a,b).

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow y = 12(x + 1)(x - 2) \text{ on } [-1, 2]$$

We know that polynomials are continuous and differentiable over R.

Let's check the values of y at the extremums

$$\Rightarrow y(-1) = 12(-1 + 1)(-1 - 2)$$

$$\Rightarrow y(-1) = 12(0)(-3)$$

$$\Rightarrow y(-1) = 0$$

$$\Rightarrow y(2) = 12(2 + 1)(2 - 2) \Rightarrow y(2) = 12(3)(0)$$

$$\Rightarrow y(2) = 0$$

We got $y(-1) = y(2)$. So, there exists a c such that $f'(c) = 0$.

For a curve g to have a tangent parallel to the x - axis at point r, the criteria to be satisfied is $g'(r) = 0$.

$$\Rightarrow y'(x) = 0$$

$$\Rightarrow \frac{d(12(x+1)(x-2))}{dx} = 0$$

$$\Rightarrow 12 \left((x+1) \frac{d(x-2)}{dx} + (x-2) \frac{d(x+1)}{dx} \right) = 0$$

$$\Rightarrow ((x+1) \times 1) + ((x-2) \times 1) = 0$$

$$\Rightarrow x + 1 + x - 2 = 0$$

$$\Rightarrow 2x - 1 = 0$$

$$\Rightarrow 2x = 1$$

$$\Rightarrow x = \frac{1}{2}$$

The value of y is

$$\Rightarrow y = 12\left(\frac{1}{2} + 1\right)\left(\frac{1}{2} - 2\right)$$

$$\Rightarrow y = 12\left(\frac{3}{2}\right)\left(-\frac{3}{2}\right)$$

$$\Rightarrow y = -27$$

The point at which the curve has tangent parallel to x - axis is $\left(\frac{1}{2}, -27\right)$.

6. Question

If $f: [-5, 5] \rightarrow \mathbb{R}$ is differentiable and if $f'(x)$ doesn't vanish anywhere, then prove that $f(-5) \neq f(5)$.

Answer

Given that f is continuous and differentiable in the interval $[-5, 5]$.

It is also given that $f'(x)$ doesn't vanish anywhere.

According to Rolle's theorem for a differentiable function on $[a, b]$ will have at least one $c \in (a, b)$ such that $f'(c) = 0$, if the following condition had satisfied:

$$\Rightarrow f(a) = f(b).$$

According to the problem it is given for any value of x , say r the values never equals to zero.

$$\Rightarrow f'(r) \neq 0$$

This is possible when Rolle's theorem is not applicable.

Let us Recap the Rolle's theorem:

For a Real valued function ' f ':

a) The function ' f ' needs to be continuous in the closed interval $[a, b]$.

b) The function ' f ' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

First, two conditions are satisfied according to the problem, so the only condition that cannot be satisfied is (c).

So, we can clearly say that $f(-5) \neq f(5)$.

7 A. Question

Examine if the Rolle's theorem applies to anyone of the following functions:

$$f(x) = [x] \text{ for } x \in [5, 9]$$

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function ' f ':

a) The function ' f ' needs to be continuous in the closed interval $[a, b]$.

b) The function ' f ' needs differentiable on the open interval (a, b) .

$$c) f(a) = f(b)$$

Then there exists at least one c in the open interval (a,b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = [x] \text{ for } x \in [5,9]$$

Let us check the continuity of the function ' f '.

Here in the interval $x \in [5,9]$, the function has to be Right continuous at $x = 5$ and left continuous at $x = 5$.

Right Hand Limit:

$$\Rightarrow \lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} [x]$$

$$\Rightarrow \lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+ + h} [x] \text{ where } h > 0.$$

$$\Rightarrow \lim_{x \rightarrow 5^+} f(x) = \lim_{h \rightarrow 0} 5$$

$$\Rightarrow \lim_{x \rightarrow 5^+} f(x) = 5 \dots\dots(1)$$

Left Hand Limit:

$$\Rightarrow \lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^-} [x]$$

$$\Rightarrow \lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^- - h} [x], \text{ where } h > 0$$

$$\Rightarrow \lim_{x \rightarrow 5^-} f(x) = \lim_{h \rightarrow 0} 4$$

$$\Rightarrow \lim_{x \rightarrow 5^-} f(x) = 4 \dots\dots(2)$$

From (1) and (2), we can clearly see that the limits are not same so, the function is not continuous in the interval $[5,9]$.

$\therefore$ Rolle's theorem is not applicable for the function f in the interval $[5,9]$.

7 B. Question

Examine if the Rolle's theorem applies to anyone of the following functions:

$$f(x) = [x] \text{ for } x \in [-2,2]$$

Answer

Given function is:

$$\Rightarrow f(x) = [x] \text{ for } x \in [-2,2]$$

Let us check the continuity of the function ' f '.

Here in the interval $x \in [-2,1]$, the function has to be Right continuous at $x = 2$ and left continuous at $x = 2$.

Right Hand Limit:

$$\Rightarrow \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} [x]$$

$$\Rightarrow \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+ + h} [x] \text{ where } h > 0.$$

$$\Rightarrow \lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} 2$$

$$\Rightarrow \lim_{x \rightarrow 2^+} f(x) = 2 \dots\dots(1)$$

Left Hand Limit:

$$\Rightarrow \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} [x]$$

$$\Rightarrow \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2-h} [x], \text{ where } h > 0$$

$$\Rightarrow \lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} 1$$

$$\Rightarrow \lim_{x \rightarrow 2^-} f(x) = 1 \dots\dots(2)$$

From (1) and (2), we can see that the limits are not the same so, the function is not continuous in the interval $[-2, 2]$.

$\therefore$ Rolle's theorem is not applicable for the function f in the interval $[-2, 2]$.

8. Question

It is given that the Rolle's theorem holds for the function $f(x) = x^3 + bx^2 + cx$, $x \in [1, 2]$ at the point $x = 4/3$. Find the values of b and c .

Answer

First, let us write the conditions for the applicability of Rolle's theorem:

For a Real valued function 'f':

a) The function 'f' needs to be continuous in the closed interval $[a, b]$.

b) The function 'f' needs differentiable on the open interval (a, b) .

c) $f(a) = f(b)$

Then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Given function is:

$$\Rightarrow f(x) = x^3 + bx^2 + cx, x \in [1, 2]$$

According to the problem the Rolle's theorem holds for the function 'f' at $x = \frac{4}{3}$.

We can say that $f'\left(\frac{4}{3}\right) = 0$.

Let's find the derivative of $f(x)$

$$\Rightarrow f'(x) = \frac{d(x^3 + bx^2 + cx)}{dx}$$

$$\Rightarrow f'(x) = \frac{d(x^3)}{dx} + \frac{d(bx^2)}{dx} + \frac{d(cx)}{dx}$$

$$\Rightarrow f'(x) = 3x^2 + 2bx + c$$

$$\text{We have } f'\left(\frac{4}{3}\right) = 0$$

$$\Rightarrow 3\left(\frac{4}{3}\right)^2 + 2b\left(\frac{4}{3}\right) + c = 0$$

$$\Rightarrow 3\left(\frac{16}{9}\right) + b\left(\frac{8}{3}\right) + c = 0$$

$$\Rightarrow \frac{16}{3} + \frac{8b}{3} + c = 0$$

$$\Rightarrow 8b + 3c = -16 \dots\dots (1)$$

We also have $f(1) = f(2)$

$$\Rightarrow (1)^3 + b(1)^2 + c(1) = (2)^3 + b(2)^2 + c(2)$$

$$\Rightarrow 1 + b(1) + c = 8 + b(4) + 2c$$

$$\Rightarrow 3b + c = -7 \dots\dots(2)$$

On solving (1) and (2), we get

$$\Rightarrow b = -5 \text{ and } c = 8$$

$\therefore$ The values of b and c is - 5 and 8.

Exercise 15.2

1 A. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x^2 - 1 \text{ on } [2, 3]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = x^2 - 1 \text{ on } [2, 3]$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[2, 3]$ and differentiable in $(2, 3)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (2, 3)$ such that:

$$f'(c) = \frac{f(3) - f(2)}{3 - 2}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(2)}{1}$$

$$f(x) = x^2 - 1$$

Differentiating with respect to x:

$$f'(x) = 2x$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2c$$

For $f(3)$, put the value of $x=3$ in $f(x)$:

$$f(3) = (3)^2 - 1$$

$$= 9 - 1$$

$$= 8$$

For $f(2)$, put the value of $x=2$ in $f(x)$:

$$f(2) = (2)^2 - 1$$

$$= 4 - 1$$

$$= 3$$

$$\therefore f'(c) = f(3) - f(2)$$

$$\Rightarrow 2c = 8 - 3$$

$$\Rightarrow 2c = 5$$

$$\Rightarrow c = \frac{5}{2} \in (2, 3)$$

Hence, Lagrange's mean value theorem is verified.

1 B. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x^3 - 2x^2 - x + 3 \text{ on } [0, 1]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = x^3 - 2x^2 - x + 3 \text{ on } [0, 1]$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 1]$ and differentiable in $(0, 1)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 1)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow f'(c) = \frac{f(1) - f(0)}{1}$$

$$f(x) = x^3 - 2x^2 - x + 3$$

Differentiating with respect to x :

$$f'(x) = 3x^2 - 2(2x) - 1$$

$$= 3x^2 - 4x - 1$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 3c^2 - 4c - 1$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = (1)^3 - 2(1)^2 - (1) + 3$$

$$= 1 - 2 - 1 + 3$$

$$= 1$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = (0)^3 - 2(0)^2 - (0) + 3$$

$$= 0 - 0 - 0 + 3$$

$$= 3$$

$$\therefore f'(c) = f(1) - f(0)$$

$$\Rightarrow 3c^2 - 4c - 1 = 1 - 3$$

$$\Rightarrow 3c^2 - 4c = 1 + 1 - 3$$

$$\Rightarrow 3c^2 - 4c = -1$$

$$\Rightarrow 3c^2 - 4c + 1 = 0$$

$$\Rightarrow 3c^2 - 3c - c + 1 = 0$$

$$\Rightarrow 3c(c - 1) - 1(c - 1) = 0$$

$$\Rightarrow (3c - 1)(c - 1) = 0$$

$$\Rightarrow c = \frac{1}{3}, 1$$

$$\Rightarrow c = \frac{1}{3} \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

1 C. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x(x - 1) \text{ on } [1, 2]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = x(x - 1) \text{ on } [1, 2]$$

$$= x^2 - x$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[1, 2]$ and differentiable in $(1, 2)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 2)$ such that:

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(1)}{1}$$

$$f(x) = x^2 - x$$

Differentiating with respect to x :

$$f'(x) = 2x - 1$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2c - 1$$

For $f(2)$, put the value of $x=2$ in $f(x)$:

$$f(2) = (2)^2 - 2$$

$$= 4 - 2$$

$$= 2$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = (1)^2 - 1$$

$$= 1 - 1$$

$$= 0$$

$$\therefore f'(c) = f(2) - f(1)$$

$$\Rightarrow 2c - 1 = 2 - 0$$

$$\Rightarrow 2c = 2 + 1$$

$$\Rightarrow 2c = 3$$

$$\Rightarrow c = \frac{3}{2} \in (1, 2)$$

Hence, Lagrange's mean value theorem is verified.

1 D. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x^2 - 3x + 2 \text{ on } [-1, 2]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = x^2 - 3x + 2 \text{ on } [-1, 2]$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[-1, 2]$ and differentiable in $(-1, 2)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (-1, 2)$ such that:

$$f'(c) = \frac{f(2) - f(-1)}{2 - (-1)}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(-1)}{2 + 1}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(-1)}{3}$$

$$f(x) = x^2 - 3x + 2$$

Differentiating with respect to x :

$$f'(x) = 2x - 3$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2c - 3$$

For $f(2)$, put the value of $x=2$ in $f(x)$:

$$\begin{aligned}f(2) &= (2)^2 - 3(2) + 2 \\&= 4 - 6 + 2 \\&= 0\end{aligned}$$

For $f(-1)$, put the value of $x=-1$ in $f(x)$:

$$\begin{aligned}f(-1) &= (-1)^2 - 3(-1) + 2 \\&= 1 + 3 + 2 \\&= 6\end{aligned}$$

$$f'(c) = \frac{f(2) - f(-1)}{3}$$

$$\Rightarrow 2c - 3 = \frac{0 - 6}{3}$$

$$\Rightarrow 2c = \frac{-6}{3} + 3$$

$$\Rightarrow 2c = -2 + 3$$

$$\Rightarrow 2c = -1$$

$$\Rightarrow c = \frac{-1}{2} \in (-1, 2)$$

Hence, Lagrange's mean value theorem is verified.

1 E. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = 2x^2 - 3x + 1 \text{ on } [1, 3]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = 2x^2 - 3x + 1 \text{ on } [1, 3]$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[1, 3]$ and differentiable in $(1, 3)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 3)$ such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$$f(x) = 2x^2 - 3x + 1$$

Differentiating with respect to x :

$$f'(x) = 2(2x) - 3$$

$$= 4x - 3$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 4c - 3$$

For $f(3)$, put the value of $x=3$ in $f(x)$:

$$f(3) = 2(3)^2 - 3(3) + 1$$

$$= 2(9) - 9 + 1$$

$$= 18 - 8 = 10$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = 2(1)^2 - 3(1) + 1$$

$$= 2(1) - 3 + 1$$

$$= 2 - 2 = 0$$

$$f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow 4c - 3 = \frac{10 - 0}{2}$$

$$\Rightarrow 4c = \frac{10}{2} + 3$$

$$\Rightarrow 4c = 5 + 3$$

$$\Rightarrow 4c = 8$$

$$\Rightarrow c = \frac{8}{4} = 2 \in (1, 3)$$

Hence, Lagrange's mean value theorem is verified.

1 F. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x^2 - 2x + 4 \text{ on } [1, 5]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = x^2 - 2x + 4 \text{ on } [1, 5]$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[1, 5]$ and differentiable in $(1, 5)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 5)$ such that:

$$f'(c) = \frac{f(5) - f(1)}{5 - 1}$$

$$\Rightarrow f'(c) = \frac{f(5) - f(1)}{4}$$

$$f(x) = x^2 - 2x + 4$$

Differentiating with respect to x:

$$f'(x) = 2x - 2$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2c - 2$$

For $f(5)$, put the value of $x=5$ in $f(x)$:

$$f(5) = (5)^2 - 2(5) + 4$$

$$= 25 - 10 + 4$$

$$= 19$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = (1)^2 - 2(1) + 4$$

$$= 1 - 2 + 4$$

$$= 3$$

$$f'(c) = \frac{f(5) - f(1)}{4}$$

$$\Rightarrow 2c - 2 = \frac{19 - 3}{4}$$

$$\Rightarrow 2c = \frac{16}{4} + 2$$

$$\Rightarrow 2c = 4 + 2$$

$$\Rightarrow 2c = 6$$

$$\Rightarrow c = \frac{6}{2} = 3 \in (1, 5)$$

Hence, Lagrange's mean value theorem is verified.

1 G. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = 2x - x^2 \text{ on } [0, 1]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = 2x - x^2 \text{ on } [0, 1]$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 1]$ and differentiable in $(0, 1)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 1)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow f'(c) = f(1) - f(0)$$

$$f(x) = 2x - x^2$$

Differentiating with respect to x :

$$f'(x) = 2 - 2x$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2 - 2c$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = 2(1) - (1)^2$$

$$= 2 - 1$$

$$= 1$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = 2(0) - (0)^2$$

$$= 0 - 0$$

$$= 0$$

$$f'(c) = f(1) - f(0)$$

$$\Rightarrow 2 - 2c = 1 - 0$$

$$\Rightarrow -2c = 1 - 2$$

$$\Rightarrow -2c = -1$$

$$\Rightarrow c = \frac{-1}{-2} = \frac{1}{2} \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

1 H. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point ' c ' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = (x - 1)(x - 2)(x - 3) \text{ on } [0, 4]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = (x - 1)(x - 2)(x - 3) \text{ on } [0, 4]$$

$$= (x^2 - x - 2x + 3)(x - 3)$$

$$= (x^2 - 3x + 3)(x - 3)$$

$$= x^3 - 3x^2 + 3x - 3x^2 + 9x - 9$$

$$= x^3 - 6x^2 + 12x - 9 \text{ on } [0, 4]$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 4]$ and differentiable in $(0, 4)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 4)$ such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$$f(x) = x^3 - 6x^2 + 12x - 9$$

Differentiating with respect to x :

$$f'(x) = 3x^2 - 6(2x) + 12$$

$$= 3x^2 - 12x + 12$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 3c^2 - 12c + 12$$

For $f(4)$, put the value of $x=4$ in $f(x)$:

$$f(4) = (4)^3 - 6(4)^2 + 12(4) - 9$$

$$= 64 - 96 + 48 - 9$$

$$= 7$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = (0)^3 - 6(0)^2 + 12(0) - 9$$

$$= 0 - 0 + 0 - 9$$

$$= -9$$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{7 - (-9)}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{7 + 9}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = \frac{16}{4}$$

$$\Rightarrow 3c^2 - 12c + 12 = 4$$

$$\Rightarrow 3c^2 - 12c + 8 = 0$$

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 8}}{2 \times 3}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{48}}{6}$$

$$\Rightarrow c = \frac{12 \pm 4\sqrt{3}}{6}$$

$$\Rightarrow c = \frac{12}{6} \pm \frac{4\sqrt{3}}{6}$$

$$\Rightarrow c = 2 \pm \frac{2\sqrt{3}}{3}$$

$$\Rightarrow c = 2 + \frac{2\sqrt{3}}{3}, 2 - \frac{2\sqrt{3}}{3} \in c$$

Hence, Lagrange's mean value theorem is verified.

1 I. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$\sqrt{25 - x^2} \text{ on } [-3, 4]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = \sqrt{25 - x^2} \text{ on } [-3, 4]$$

Here,

$$\sqrt{25 - x^2} > 0$$

$$\Rightarrow 25 - x^2 > 0$$

$$\Rightarrow x^2 < 25$$

$$\Rightarrow -5 < x < 5$$

$$\Rightarrow \sqrt{25 - x^2} \text{ has unique values for all } x \in (-5, 5)$$

$\therefore f(x)$ is continuous in $[-3, 4]$

$$f(x) = (25 - x^2)^{\frac{1}{2}}$$

Differentiating with respect to x :

$$f'(x) = \frac{1}{2} (25 - x^2)^{\left(\frac{1}{2} - 1\right)} \frac{d(25 - x^2)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2} (25 - x^2)^{-\frac{1}{2}} (-2x)$$

$$\Rightarrow f'(x) = \frac{-2x}{2 (25 - x^2)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{-2x}{2 (25 - x^2)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

Here also,

$$\sqrt{25 - x^2} > 0$$

$$\Rightarrow -5 < x < 5$$

$\therefore f(x)$ is differentiable in $(-5, 5)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (-5, 5)$ such that:

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow f'(c) = \frac{f(5) - f(-5)}{10}$$

$$\Rightarrow f'(c) = \frac{f(5) - f(-5)}{10}$$

$$f(x) = (25 - x^2)^{\frac{1}{2}}$$

On differentiating with respect to x :

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}}$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = \frac{-c}{\sqrt{25 - c^2}}$$

For $f(5)$, put the value of $x=5$ in $f(x)$:

$$f(5) = (25 - 5^2)^{\frac{1}{2}}$$

$$\Rightarrow f(5) = (25 - 25)^{\frac{1}{2}}$$

$$\Rightarrow f(5) = (0)^{\frac{1}{2}}$$

$$\Rightarrow f(5) = 0$$

For $f(-5)$, put the value of $x=-5$ in $f(x)$:

$$f(-5) = (25 - (-5)^2)^{\frac{1}{2}}$$

$$\Rightarrow f(-5) = (25 - 25)^{\frac{1}{2}}$$

$$\Rightarrow f(-3) = (16)^{\frac{1}{2}}$$

$$\Rightarrow f(-3) = 4$$

$$f'(c) = \frac{f(4) - f(-3)}{7}$$

$$\Rightarrow \frac{-c}{\sqrt{25-c^2}} = \frac{3-4}{7}$$

$$\Rightarrow \frac{-c}{\sqrt{25-c^2}} = \frac{-1}{7}$$

$$\Rightarrow -7c = -\sqrt{25-c^2}$$

Squaring both sides:

$$\Rightarrow (-7c)^2 = (-\sqrt{25-c^2})^2$$

$$\Rightarrow 49c^2 = 25 - c^2$$

$$\Rightarrow 50c^2 = 25$$

$$\Rightarrow c^2 = \frac{25}{50}$$

$$\Rightarrow c^2 = \frac{1}{2}$$

$$\Rightarrow c = \pm \frac{1}{\sqrt{2}} \in (-3, 4)$$

Hence, Lagrange's mean value theorem is verified.

1 J. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = \tan^{-1} x \text{ on } [0, 1]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b-a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = \tan^{-1} x \text{ on } [0, 1]$$

$\tan^{-1} x$ has unique value for all x between 0 and 1.

$\therefore f(x)$ is continuous in $[0, 1]$

$$f(x) = \tan^{-1} x$$

Differentiating with respect to x :

$$f'(x) = \frac{1}{1+x^2}$$

x^2 always has value greater than 0.

$$\Rightarrow 1 + x^2 > 0$$

$\therefore f(x)$ is differentiable in $(0, 1)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (-3, 4)$ such that:

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\Rightarrow f'(c) = f(1) - f(0)$$

$$f(x) = \tan^{-1} x$$

Differentiating with respect to x :

$$f'(x) = \frac{1}{1+x^2}$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = \frac{1}{1+c^2}$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = \tan^{-1} 1$$

$$\Rightarrow f(1) = \frac{\pi}{4}$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = \tan^{-1} 0$$

$$\Rightarrow f(0) = 0$$

$$f'(c) = f(1) - f(0)$$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4} - 0$$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\pi}{4}$$

$$\Rightarrow 4 = \pi(1+c^2)$$

$$\Rightarrow 4 = \pi + \pi c^2$$

$$\Rightarrow -\pi c^2 = \pi - 4$$

$$\Rightarrow c^2 = \frac{\pi - 4}{-\pi}$$

$$\Rightarrow c^2 = \frac{4 - \pi}{\pi}$$

$$\Rightarrow c = \sqrt{\frac{4}{\pi} - 1} \approx 0.52 \in (0, 1)$$

Hence, Lagrange's mean value theorem is verified.

1 K. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point ' c ' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x + \frac{1}{x} \text{ on } [1, 3]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = x + \frac{1}{x} \text{ on } [1, 3]$$

$f(x)$ has unique values for all $x \in (1, 3)$

$\therefore f(x)$ is continuous in $[1, 3]$

$$f(x) = x + \frac{1}{x} \text{ on } [1, 3]$$

Differentiating with respect to x :

$$f'(x) = 1 + (-1)(x)^{-2}$$

$$\Rightarrow f'(x) = 1 - \frac{1}{x^2}$$

$$\Rightarrow f'(x) = \frac{x^2 - 1}{x^2}$$

Here,

$$x^2 \neq 0$$

$\Rightarrow f'(x)$ exists for all values except 0

$\therefore f(x)$ is differentiable in $(1, 3)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 3)$ such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$$f(x) = x + \frac{1}{x}$$

On differentiating with respect to x :

$$f'(x) = \frac{x^2 - 1}{x^2}$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = \frac{c^2 - 1}{c^2}$$

For $f(3)$, put the value of $x=3$ in $f(x)$:

$$f(3) = 3 + \frac{1}{3}$$

$$\Rightarrow f(3) = \frac{9+1}{3}$$

$$\Rightarrow f(3) = \frac{10}{3}$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = 1 + \frac{1}{1}$$

$$\Rightarrow f(1) = 2$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow \frac{c^2 - 1}{c^2} = \frac{\frac{10}{3} - 2}{2}$$

$$\Rightarrow 2(c^2 - 1) = c^2 \left(\frac{10}{3} - 2 \right)$$

$$\Rightarrow 2(c^2 - 1) = c^2 \left(\frac{10 - 6}{3} \right)$$

$$\Rightarrow 2(c^2 - 1) = c^2 \left(\frac{4}{3} \right)$$

$$\Rightarrow 6(c^2 - 1) = 4c^2$$

$$\Rightarrow 6c^2 - 6 = 4c^2$$

$$\Rightarrow 6c^2 - 4c^2 = 6$$

$$\Rightarrow 2c^2 = 6$$

$$\Rightarrow c^2 = \frac{6}{2}$$

$$\Rightarrow c^2 = 3$$

$$\Rightarrow c = \pm\sqrt{3} \in (-3, 4)$$

Hence, Lagrange's mean value theorem is verified.

1 L. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x(x + 4)^2 \text{ on } [0, 4]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = x(x + 4)^2 \text{ on } [0, 4]$$

$$= x [(x)^2 + 2(4)(x) + (4)^2]$$

$$= x(x^2 + 8x + 16)$$

$$= x^3 + 8x^2 + 16x \text{ on } [0, 4]$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 4]$ and differentiable in $(0, 4)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 4)$ such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$$f(x) = x^3 + 8x^2 + 16x$$

Differentiating with respect to x :

$$f'(x) = 3x^2 + 8(2x) + 16$$

$$= 3x^2 + 16x + 16$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 3c^2 + 16c + 16$$

For $f(4)$, put the value of $x=4$ in $f(x)$:

$$f(4) = (4)^3 + 8(4)^2 + 16(4)$$

$$= 64 + 128 + 64$$

$$= 256$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = (0)^3 + 8(0)^2 + 16(0)$$

$$= 0 + 0 + 0$$

$$= 0$$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = \frac{256 - 0}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = \frac{256}{4}$$

$$\Rightarrow 3c^2 + 16c + 16 = 64$$

$$\Rightarrow 3c^2 + 16c + 16 - 64 = 0$$

$$\Rightarrow 3c^2 + 16c - 48 = 0$$

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow c = \frac{-(16) \pm \sqrt{(16)^2 - 4 \times 3 \times (-48)}}{2 \times 3}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{256 + 576}}{6}$$

$$\Rightarrow c = \frac{-16 \pm \sqrt{832}}{6}$$

$$\Rightarrow c = \frac{-16 \pm 8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-16}{6} \pm \frac{8\sqrt{13}}{6}$$

$$\Rightarrow c = \frac{-8}{3} \pm \frac{4\sqrt{13}}{3}$$

$$\Rightarrow c = \frac{-8}{3} + \frac{4\sqrt{13}}{3}, \frac{-8}{3} - \frac{4\sqrt{13}}{3} \in c$$

Hence, Lagrange's mean value theorem is verified.

1 M. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x\sqrt{x^2 - 4} \text{ on } [2, 4]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = \sqrt{x^2 - 4} \text{ on } [2, 4]$$

Here,

$$\sqrt{x^2 - 4} > 0$$

$$\Rightarrow x^2 - 4 > 0$$

$$\Rightarrow x^2 > 4$$

$$\Rightarrow f(x) \text{ exists for all values except } (-2, 2)$$

$$\therefore f(x) \text{ is continuous in } [2, 4]$$

$$f(x) = \sqrt{x^2 - 4}$$

Differentiating with respect to x :

$$f'(x) = \frac{1}{2}(x^2 - 4)^{\left(\frac{1}{2} - 1\right)} \frac{d(x^2 - 4)}{dx}$$

$$\Rightarrow f'(x) = \frac{1}{2}(x^2 - 4)^{-\frac{1}{2}}(2x)$$

$$\Rightarrow f'(x) = \frac{2x}{2(x^2 - 4)^{\frac{1}{2}}}$$

$$\Rightarrow f'(x) = \frac{x}{\sqrt{x^2 - 4}}$$

Here also,

$$\sqrt{x^2 - 4} > 0$$

$\Rightarrow f'(x)$ exists for all values of x except $(2, -2)$

$\therefore f(x)$ is differentiable in $(2, 4)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (2, 4)$ such that:

$$f'(c) = \frac{f(4) - f(2)}{4 - 2}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(2)}{2}$$

$$f(x) = \sqrt{x^2 - 4}$$

On differentiating with respect to x :

$$f'(x) = \frac{x}{\sqrt{x^2 - 4}}$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = \frac{c}{\sqrt{c^2 - 4}}$$

For $f(4)$, put the value of $x=4$ in $f(x)$:

$$f(4) = \sqrt{4^2 - 4}$$

$$\Rightarrow f(4) = (16 - 4)^{\frac{1}{2}}$$

$$\Rightarrow f(4) = \sqrt{12}$$

$$\Rightarrow f(4) = 2\sqrt{3}$$

For $f(2)$, put the value of $x=2$ in $f(x)$:

$$f(2) = \sqrt{2^2 - 4}$$

$$\Rightarrow f(2) = (4 - 4)^{\frac{1}{2}}$$

$$\Rightarrow f(2) = 0$$

$$\Rightarrow f'(c) = \frac{f(4) - f(2)}{2}$$

$$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \frac{2\sqrt{3} - 0}{2}$$

$$\Rightarrow \frac{c}{\sqrt{c^2 - 4}} = \sqrt{3}$$

$$\Rightarrow c = (\sqrt{3})\sqrt{c^2 - 4}$$

Squaring both sides:

$$\Rightarrow (c)^2 = ((\sqrt{3})\sqrt{c^2 - 4})^2$$

$$\Rightarrow c^2 = 3(c^2 - 4)$$

$$\Rightarrow c^2 = 3c^2 - 12$$

$$\Rightarrow -2c^2 = -12$$

$$\Rightarrow c^2 = \frac{-12}{-2}$$

$$\Rightarrow c^2 = 6$$

$$\Rightarrow c = \pm\sqrt{6}$$

$$\Rightarrow c = \sqrt{6} \in (2, 4)$$

Hence, Lagrange's mean value theorem is verified.

1 N. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x^2 + x - 1 \text{ on } [0, 4]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = x^2 + x - 1 \text{ on } [0, 4]$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 4]$ and differentiable in $(0, 4)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 4)$ such that:

$$f'(c) = \frac{f(4) - f(0)}{4 - 0}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4}$$

$$f(x) = x^2 + x - 1$$

Differentiating with respect to x :

$$f'(x) = 2x + 1$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2c + 1$$

For $f(4)$, put the value of $x=4$ in $f(x)$:

$$f(4) = (4)^2 + 4 - 1$$

$$= 16 + 4 - 1$$

$$= 19$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = (0)^2 + 0 - 1$$

$$= 0 + 0 - 1$$

$$= -1$$

$$f'(c) = \frac{f(4) - f(0)}{4}$$

$$\Rightarrow 2c + 1 = \frac{19 - (-1)}{4}$$

$$\Rightarrow 2c + 1 = \frac{20}{4}$$

$$\Rightarrow 2c + 1 = 5$$

$$\Rightarrow 2c = 5 - 1$$

$$\Rightarrow 2c = 4$$

$$\Rightarrow c = \frac{4}{2} = 2 \in (0, 4)$$

Hence, Lagrange's mean value theorem is verified.

1 O. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = \sin x - \sin 2x - x \text{ on } [0, \pi]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = \sin x - \sin 2x - x \text{ on } [0, \pi]$$

$\sin x$ and $\cos x$ functions are **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, \pi)$ such that:

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0}$$

$$\Rightarrow f'(c) = \frac{f(\pi) - f(0)}{\pi}$$

$$f(x) = \sin x - \sin 2x - x$$

Differentiating with respect to x:

$$f(x) = \sin x - \sin 2x - x$$

$$\Rightarrow f'(x) = \cos x - \cos 2x \frac{d(2x)}{dx} - 1$$

$$\Rightarrow f'(x) = \cos x - 2\cos 2x - 1$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = \cos c - 2\cos 2c - 1$$

For $f(\pi)$, put the value of $x=\pi$ in $f(x)$:

$$f(\pi) = \sin \pi - \sin 2\pi - \pi$$

$$= 0 - 0 - \pi$$

$$= -\pi$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = \sin 0 - \sin 2(0) - 0$$

$$= \sin 0 - \sin 0 - 0$$

$$= 0 - 0 - 0$$

$$= 0$$

$$f'(c) = \frac{f(\pi) - f(0)}{\pi}$$

$$\Rightarrow \cos c - 2\cos 2c - 1 = \frac{-\pi - 0}{\pi}$$

$$\Rightarrow \cos c - 2\cos 2c - 1 = -1$$

$$\Rightarrow \cos c - 2(2\cos^2 c - 1) = -1 + 1$$

$$\Rightarrow \cos c - 4\cos^2 c + 2 = 0$$

$$\Rightarrow 4\cos^2 c - \cos c - 2 = 0$$

For quadratic equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \cos c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 4 \times (-2)}}{2 \times 4}$$

$$\Rightarrow \cos c = \frac{1 \pm \sqrt{1 + 32}}{8}$$

$$\Rightarrow \cos c = \frac{1 \pm \sqrt{33}}{8}$$

$$\Rightarrow c = \cos^{-1} \left(\frac{1 \pm \sqrt{33}}{8} \right) \in (0, \pi)$$

Hence, Lagrange's mean value theorem is verified.

1 P. Question

Verify Lagrange's mean value theorem for the following functions on the indicated intervals. In each find a point 'c' in the indicated interval as stated by the Lagrange's mean value theorem :

$$f(x) = x^3 - 5x^2 - 3x \text{ on } [1, 3]$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b)-f(a)=f'(c)(b-a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = x^3 - 5x^2 - 3x \text{ on } [1, 3]$$

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[1, 3]$ and differentiable in $(1, 3)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 3)$ such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$$f(x) = x^3 - 5x^2 - 3x$$

Differentiating with respect to x :

$$f'(x) = 3x^2 - 5(2x) - 3$$

$$= 3x^2 - 10x - 3$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 3c^2 - 10c - 3$$

For $f(3)$, put the value of $x=3$ in $f(x)$:

$$f(3) = (3)^3 - 5(3)^2 - 3(3)$$

$$= 27 - 45 - 9$$

$$= -27$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = (1)^3 - 5(1)^2 - 3(1)$$

$$= 1 - 5 - 3$$

$$= -7$$

$$f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{(-27) - (-7)}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{-27+7}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = \frac{-20}{2}$$

$$\Rightarrow 3c^2 - 10c - 3 = -10$$

$$\Rightarrow 3c^2 - 10c - 3 + 10 = 0$$

$$\Rightarrow 3c^2 - 10c + 7 = 0$$

$$\Rightarrow 3c^2 - 7c - 3c + 7 = 0$$

$$\Rightarrow c(3c - 7) - 1(3c - 7) = 0$$

$$\Rightarrow (3c - 7)(c - 1) = 0$$

$$\Rightarrow c = \frac{7}{3}, 1$$

$$\Rightarrow c = \frac{7}{3} \in (1, 3)$$

Hence, Lagrange's mean value theorem is verified.

2. Question

Discuss the applicability of Lagrange's mean value theorem for the function $f(x) = |x|$ on $[-1, 1]$.

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = |x| \text{ on } [-1, 1]$$

$$\text{So } f(x) \text{ can be defined as } = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$

For differentiability at $x=0$,

$$\text{LHD} = \lim_{x \rightarrow 0^-} \frac{f(0-h) - f(0)}{-h}$$

{Since $f(x) = -x$, $x < 0$ }

$$= \lim_{x \rightarrow 0^-} \frac{-(0-h) - 0}{-h}$$

$$= \lim_{x \rightarrow 0^-} \frac{h - 0}{-h}$$

$$= \lim_{x \rightarrow 0^-} \frac{h}{-h}$$

$$= -1$$

$$\text{RHD} = \lim_{x \rightarrow 0^+} \frac{f(0-h) - f(0)}{-h}$$

{Since $f(x) = x$, $x > 0$ }

$$= \lim_{x \rightarrow 0^+} \frac{(0-h) - 0}{-h}$$

$$= \lim_{x \rightarrow 0^-} \frac{-h-0}{-h}$$

$$= \lim_{x \rightarrow 0^-} \frac{-h}{-h}$$

$$= 1$$

LHD $\neq$ RHD

$\Rightarrow f(x)$ is not differential at $x=0$

$\therefore$ Lagrange's mean value theorem is not applicable for the function $f(x) = |x|$ on $[-1, 1]$.

3. Question

Show that the Lagrange's mean value theorem is not applicable to the function $f(x) = 1/x$ on $[-1, 1]$.

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = \frac{1}{x} \text{ on } [-1, 1]$$

Here,

$$x \neq 0$$

$\Rightarrow f(x)$ exists for all values of x except 0

$\Rightarrow f(x)$ is discontinuous at $x=0$

$\therefore f(x)$ is not continuous in $[-1, 1]$

Hence the lagrange's mean value theorem is not applicable to the

$$\text{function } f(x) = \frac{1}{x} \text{ on } [-1, 1]$$

4. Question

"Verify the hypothesis and conclusion of Lagrange's mean value theorem for the function

$$f(x) = \frac{1}{4x-1}, 1 < x < 4.$$

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$\text{Let } f(x) = \frac{1}{4x-1} \text{ on } [1, 4]$$

$$4x - 1 > 0$$

$\Rightarrow f(x)$ has unique values for all x except $\frac{1}{4}$

$\therefore f(x)$ is continuous in $[1, 4]$

$$f(x) = \frac{1}{4x - 1}$$

Differentiating with respect to x :

$$f'(x) = (-1)(4x - 1)^{-2}(4)$$

$$\Rightarrow f'(x) = -\frac{4}{(4x - 1)^2}$$

Here,

$$\Rightarrow 4x - 1 > 0$$

$\Rightarrow f'(x)$ has unique values for all x except $\frac{1}{4}$

$\therefore f(x)$ is differentiable in $(1, 4)$

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 4)$ such that:

$$f'(c) = \frac{f(4) - f(1)}{4 - 1}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(1)}{3}$$

$$f(x) = \frac{1}{4x - 1}$$

On differentiating with respect to x :

$$f'(x) = -\frac{4}{(4x - 1)^2}$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = -\frac{4}{(4c - 1)^2}$$

For $f(4)$, put the value of $x=4$ in $f(x)$:

$$f(4) = \frac{1}{4(4) - 1}$$

$$\Rightarrow f(4) = \frac{1}{16 - 1}$$

$$\Rightarrow f(4) = \frac{1}{15}$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = \frac{1}{4(1) - 1}$$

$$\Rightarrow f(1) = \frac{1}{4 - 1}$$

$$\Rightarrow f(1) = \frac{1}{3}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(1)}{3}$$

$$\Rightarrow -\frac{4}{(4c-1)^2} = \frac{\frac{1}{15} - \frac{1}{3}}{3}$$

$$\Rightarrow -3(4) = (4c-1)^2 \left(\frac{1}{15} - \frac{1}{3} \right)$$

$$\Rightarrow -12 = (4c-1)^2 \left(\frac{3-15}{45} \right)$$

$$\Rightarrow -12 = (4c-1)^2 \left(\frac{-12}{45} \right)$$

$$\Rightarrow -12 \times \frac{45}{-12} = (4c-1)^2$$

$$\Rightarrow (4c-1)^2 = 45$$

$$\Rightarrow (4c-1) = \pm\sqrt{45}$$

$$\Rightarrow (4c-1) = \pm 3\sqrt{5}$$

$$\Rightarrow c = \frac{\pm 3\sqrt{5} + 1}{4}$$

$$\Rightarrow c = \frac{3\sqrt{5}+1}{4} \approx 1.92 \in (1, 4)$$

Hence, Lagrange's mean value theorem is verified.

5. Question

Find a point on the parabola $y = (x-4)^2$, where the tangent is parallel to the chord joining (4, 0) and (5, 1).

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b-a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}$$

This theorem is also known as First Mean Value Theorem.

Let $f(x) = (x-4)^2$ on $[4, 5]$

This interval $[a, b]$ is obtained by x - coordinates of the points of the chord.

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[4, 5]$ and differentiable in $(4, 5)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (4, 5)$ such that:

$$f'(c) = \frac{f(5) - f(4)}{5-4}$$

$$\Rightarrow f'(c) = \frac{f(5) - f(4)}{1}$$

$$f(x) = (x-4)^2$$

Differentiating with respect to x:

$$f'(x) = 2(x - 4) \frac{d(x - 4)}{dx}$$

$$\Rightarrow f'(x) = 2(x - 4)(1)$$

$$\Rightarrow f'(x) = 2(x - 4)$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2(c - 4)$$

For $f(5)$, put the value of $x=5$ in $f(x)$:

$$f(5) = (5 - 4)^2$$

$$= (1)^2$$

$$= 1$$

For $f(4)$, put the value of $x=4$ in $f(x)$:

$$f(4) = (4 - 4)^2$$

$$= (0)^2$$

$$= 0$$

$$f'(c) = f(5) - f(4)$$

$$\Rightarrow 2(c - 4) = 1 - 0$$

$$\Rightarrow 2c - 8 = 1$$

$$\Rightarrow 2c = 1 + 8$$

$$\Rightarrow c = \frac{9}{2} = 4.5 \in (4, 5)$$

We know that, the value of c obtained in Lagrange's Mean value Theorem is nothing but the value of x - coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points $(4, 0)$ and $(5, 1)$.

Now, Put this value of x in $f(x)$ to obtain y :

$$y = (x - 4)^2$$

$$\Rightarrow y = \left(\frac{9}{2} - 4\right)^2$$

$$\Rightarrow y = \left(\frac{9-8}{2}\right)^2$$

$$\Rightarrow y = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow y = \frac{1}{4}$$

Hence, the required point is $\left(\frac{9}{2}, \frac{1}{4}\right)$

6. Question

Find a point on the curve $y = x^2 + x$, where the tangent is parallel to the chord joining $(0, 0)$ and $(1, 2)$.

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b)-f(a)=f'(c)(b-a)$$

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}$$

This theorem is also known as First Mean Value Theorem.

$$\text{Let } f(x) = x^2 + x \text{ on } [0, 1]$$

This interval $[a, b]$ is obtained by x - coordinates of the points of the chord.

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[0, 1]$ and differentiable in $(0, 1)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (0, 1)$ such that:

$$f'(c)=\frac{f(1)-f(0)}{1-0}$$

$$\Rightarrow f'(c)=\frac{f(1)-f(0)}{1}$$

$$f(x) = x^2 + x$$

Differentiating with respect to x :

$$f'(x) = 2x + 1$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2c + 1$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = (1)^2 + 1$$

$$= 1 + 1$$

$$= 2$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = (0)^2 + 0$$

$$= 0 + 0$$

$$= 0$$

$$f'(c) = f(1) - f(0)$$

$$\Rightarrow 2c + 1 = 2 - 0$$

$$\Rightarrow 2c = 2 - 1$$

$$\Rightarrow 2c = 1$$

$$\Rightarrow c = \frac{1}{2} = 0.5 \in (0, 1)$$

We know that the value of c obtained in Lagrange's Mean value Theorem is nothing but the value of x - coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points $(0, 0)$ and $(1, 2)$.

Now, put this value of x in $f(x)$ to obtain y :

$$y = x^2 + x$$

$$\Rightarrow y = \left(\frac{1}{2}\right)^2 + \frac{1}{2}$$

$$\Rightarrow y = \frac{1}{4} + \frac{1}{2}$$

$$\Rightarrow y = \frac{1+2}{4}$$

$$\Rightarrow y = \frac{3}{4}$$

Hence, the required point is $\left(\frac{1}{2}, \frac{3}{4}\right)$

7. Question

Find a point on the parabola $y = (x - 3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1).

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

Let $f(x) = (x - 3)^2$ on $[3, 4]$

This interval $[a, b]$ is obtained by x - coordinates of the points of the chord.

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[3, 4]$ and differentiable in $(3, 4)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (4, 5)$ such that:

$$f'(c) = \frac{f(4) - f(3)}{4 - 3}$$

$$\Rightarrow f'(c) = \frac{f(4) - f(3)}{1}$$

$$f(x) = (x - 3)^2$$

Differentiating with respect to x :

$$f'(x) = 2(x - 3) \frac{d(x - 3)}{dx}$$

$$\Rightarrow f'(x) = 2(x - 3)(1)$$

$$\Rightarrow f'(x) = 2(x - 3)$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 2(c - 3)$$

For $f(4)$, put the value of $x=4$ in $f(x)$:

$$f(4) = (4 - 3)^2$$

$$= (1)^2$$

$$= 1$$

For $f(3)$, put the value of $x=3$ in $f(x)$:

$$f(3) = (3 - 3)^2$$

$$= (0)^2$$

$$= 0$$

$$f'(c) = f(4) - f(3)$$

$$\Rightarrow 2(c - 3) = 1 - 0$$

$$\Rightarrow 2c - 6 = 1$$

$$\Rightarrow 2c = 1 + 6$$

$$\Rightarrow c = \frac{7}{2} = 3.5 \in (3, 4)$$

We know that, the value of c obtained in Lagrange's Mean value Theorem is nothing but the value of x - coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points $(3, 0)$ and $(4, 1)$.

Now, Put this value of x in $f(x)$ to obtain y :

$$y = (x - 3)^2$$

$$\Rightarrow y = \left(\frac{7}{2} - 3\right)^2$$

$$\Rightarrow y = \left(\frac{7-6}{2}\right)^2$$

$$\Rightarrow y = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow y = \frac{1}{4}$$

Hence, the required point is $\left(\frac{7}{2}, \frac{1}{4}\right)$

8. Question

Find points on the curve $y = x^3 - 3x$, where the tangent to the curve is parallel to the chord joining $(1, -2)$ and $(2, 2)$.

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

Let $f(x) = x^3 - 3x$ on $[1, 2]$

This interval $[a, b]$ is obtained by x - coordinates of the points of the chord.

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[1, 2]$ and differentiable in $(1, 2)$. So both the

necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 2)$ such that:

$$f'(c) = \frac{f(2) - f(1)}{2 - 1}$$

$$\Rightarrow f'(c) = \frac{f(2) - f(1)}{1}$$

$$f(x) = x^3 - 3x$$

Differentiating with respect to x:

$$f'(x) = 3x^2 - 3$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 3c^2 - 3$$

For $f(2)$, put the value of $x=2$ in $f(x)$:

$$f(2) = (2)^3 - 3(2)$$

$$= 8 - 6$$

$$= 2$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = (1)^3 - 3(1)$$

$$= 1 - 3$$

$$= -2$$

$$f'(c) = f(2) - f(1)$$

$$\Rightarrow 3c^2 - 3 = 2 - (-2)$$

$$\Rightarrow 3c^2 - 3 = 2 + 2$$

$$\Rightarrow 3c^2 = 4 + 3$$

$$\Rightarrow c^2 = \frac{7}{3}$$

$$\Rightarrow c = \pm \sqrt{\frac{7}{3}}$$

$$\Rightarrow c = \sqrt{\frac{7}{3}} \in (1, 2)$$

We know that, the value of c obtained in Lagrange's Mean value Theorem is nothing but the value of x - coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points $(1, -2)$ and $(2, 2)$.

Now, Put this value of x in $f(x)$ to obtain y :

$$y = x^3 - 3x$$

$$\Rightarrow y = \left(\sqrt{\frac{7}{3}}\right)^3 - 3\left(\sqrt{\frac{7}{3}}\right)$$

$$\Rightarrow y = \frac{7}{3} \left(\sqrt{\frac{7}{3}} \right) - 3 \left(\sqrt{\frac{7}{3}} \right)$$

$$\Rightarrow y = \left(\sqrt{\frac{7}{3}} \right) \left(\frac{7}{3} - 3 \right)$$

$$\Rightarrow y = \left(\sqrt{\frac{7}{3}} \right) \left(\frac{7-9}{3} \right)$$

$$\Rightarrow y = \frac{-2}{3} \left(\sqrt{\frac{7}{3}} \right)$$

Hence, the required point is $\left(\sqrt{\frac{7}{3}}, \frac{-2}{3} \left(\sqrt{\frac{7}{3}} \right) \right)$

9. Question

Find a point on the curve $y = x^3 + 1$ where the tangent is parallel to the chord joining (1, 2) and (3, 28).

Answer

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

Let $f(x) = x^3 + 1$ on $[1, 3]$

This interval $[a, b]$ is obtained by x - coordinates of the points of the chord.

Every polynomial function is **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

Here, $f(x)$ is a polynomial function. So it is continuous in $[1, 3]$ and differentiable in $(1, 3)$. So both the necessary conditions of Lagrange's mean value theorem is satisfied.

Therefore, there exists a point $c \in (1, 3)$ such that:

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow f'(c) = \frac{f(3) - f(1)}{2}$$

$$f(x) = x^3 + 1$$

Differentiating with respect to x :

$$f'(x) = 3x^2$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = 3c^2$$

For $f(3)$, put the value of $x=3$ in $f(x)$:

$$f(3) = (3)^3 + 1$$

$$= 27 + 3$$

$$= 30$$

For $f(1)$, put the value of $x=1$ in $f(x)$:

$$f(1) = (1)^3 + 3$$

$$= 1 + 3$$

$$= 4$$

$$f'(c) = \frac{f(3) - f(1)}{2}$$

$$\Rightarrow 3c^2 = \frac{30 - 4}{2}$$

$$\Rightarrow 3c^2 = \frac{26}{2}$$

$$\Rightarrow 3c^2 = 13$$

$$\Rightarrow c^2 = \frac{13}{3}$$

$$\Rightarrow c = \pm \sqrt{\frac{13}{3}}$$

$$\Rightarrow c = \sqrt{\frac{13}{3}} \in (1, 3)$$

We know that, the value of c obtained in Lagrange's Mean value Theorem is nothing but the value of x - coordinate of the point of the contact of the tangent to the curve which is parallel to the chord joining the points $(1, 2)$ and $(3, 28)$.

Now, Put this value of x in $f(x)$ to obtain y :

$$y = x^3 + 1$$

$$\Rightarrow y = \left(\sqrt{\frac{13}{3}} \right)^3 + 1$$

$$\text{Hence, the required point is } \left(\sqrt{\frac{13}{3}}, \left(\sqrt{\frac{13}{3}} \right)^3 + 1 \right)$$

10. Question

Let C be a curve defined parametrically as $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $0 \leq \theta \leq \pi/2$. Determine a point P on C , where the tangent to C is parallel to the chord joining the points $(a, 0)$ and $(0, a)$.

Answer

$\sin x$ and $\cos x$ functions are **continuous** everywhere on $(-\infty, \infty)$ and **differentiable** for all arguments.

So both the necessary conditions of Lagrange's mean value theorem is satisfied.

$$x = a \cos^3 \theta$$

$$\Rightarrow \cos^3 \theta = \frac{x}{a}$$

$$\Rightarrow \cos \theta = \left(\frac{x}{a}\right)^{\frac{1}{3}}$$

$$y = a \sin^3 \theta$$

$$\Rightarrow \sin^3 \theta = \frac{y}{a}$$

$$\Rightarrow \sin \theta = \left(\frac{y}{a}\right)^{\frac{1}{3}}$$

We know that,

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\therefore \left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{a}\right)^{\frac{2}{3}} = 1$$

$$\Rightarrow (x)^{\frac{2}{3}} + (y)^{\frac{2}{3}} = (a)^{\frac{2}{3}}$$

$$\Rightarrow (y)^{\frac{2}{3}} = (a)^{\frac{2}{3}} - (x)^{\frac{2}{3}}$$

$$\Rightarrow y = \left[(a)^{\frac{2}{3}} - (x)^{\frac{2}{3}} \right]^{\frac{3}{2}}$$

$$\text{Let } f(x) = \left[(a)^{\frac{2}{3}} - (x)^{\frac{2}{3}} \right]^{\frac{3}{2}}$$

Therefore, there exists a point $c \in (0, a)$ such that:

$$f'(c) = \frac{f(a) - f(0)}{a - 0}$$

$$\Rightarrow f'(c) = \frac{f(a) - f(0)}{a}$$

$$x = a \cos^3 \theta$$

$$\Rightarrow \frac{dx}{d\theta} = \frac{d(a \cos^3 \theta)}{d\theta}$$

$$\Rightarrow \frac{dx}{d\theta} = 3a \cos^2 \theta \times \frac{d(\cos \theta)}{d\theta}$$

$$\Rightarrow \frac{dx}{d\theta} = 3a \cos^2 \theta \times (-\sin \theta)$$

$$y = a \sin^3 \theta$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{d(a \sin^3 \theta)}{d\theta}$$

$$\Rightarrow \frac{dy}{d\theta} = 3a \sin^2 \theta \times \frac{d(\sin \theta)}{d\theta}$$

$$\Rightarrow \frac{dy}{d\theta} = 3a \sin^2 \theta \times (\cos \theta)$$

$$\frac{dy}{dx} = \left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} \right)$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{3a \sin^2 \theta \times (\cos \theta)}{3a \cos^2 \theta \times (-\sin \theta)} \right)$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = -\tan \theta$$

For $f'(c)$, put the value of $x=c$ in $f'(x)$:

$$f'(c) = -\tan \theta$$

$$f(x) = \left[(a)^{\frac{2}{3}} - (x)^{\frac{2}{3}} \right]^{\frac{3}{2}}$$

For $f(a)$, put the value of $x=a$ in $f(x)$:

$$f(a) = \left[(a)^{\frac{2}{3}} - (a)^{\frac{2}{3}} \right]^{\frac{3}{2}}$$

$$= 0$$

For $f(0)$, put the value of $x=0$ in $f(x)$:

$$f(0) = \left[(a)^{\frac{2}{3}} - (0)^{\frac{2}{3}} \right]^{\frac{3}{2}}$$

$$= \left[(a)^{\frac{2}{3}} \right]^{\frac{3}{2}}$$

$$= a$$

$$f'(c) = \frac{f(a) - f(0)}{a}$$

$$\Rightarrow -\tan \theta = \frac{0 - a}{a}$$

$$\Rightarrow -\tan \theta = \frac{-a}{a}$$

$$\Rightarrow -\tan \theta = -1$$

$$\Rightarrow \tan \theta = 1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Now put the value of θ in the function of x and y :

$$x = a \cos^3 \theta$$

$$\Rightarrow x = a \cos^3 \left(\frac{\pi}{4} \right)$$

$$\Rightarrow x = a \left(\frac{1}{\sqrt{2}} \right)^3$$

$$\Rightarrow x = \frac{a}{2\sqrt{2}}$$

Similarly,

$$y = a \sin^3 \theta$$

$$\Rightarrow y = a \sin^3 \left(\frac{\pi}{4} \right)$$

$$\Rightarrow y = a \left(\frac{1}{\sqrt{2}} \right)^3$$

$$\Rightarrow y = \frac{a}{2\sqrt{2}}$$

So the required point is $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}} \right)$.

11. Question

Using Lagrange's mean value theorem, prove that

$(b - a) \sec^2 a < \tan b - \tan a < (b - a) \sec^2 b$, where $0 < a < b < \pi/2$.

Answer

Let $f(x) = \tan x$ on $[a, b]$

We know that, $\tan x$ function is **continuous** and **differentiable** on

$\left(0, \frac{\pi}{2} \right)$. Since a and b lie between 0 and $\frac{\pi}{2}$, $\tan x$ is continuous and

differentiable on (a, b) .

Lagrange's mean value theorem states that if a function $f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) , then there is at least one point $x=c$ on this interval, such that

$$f(b) - f(a) = f'(c)(b - a)$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

This theorem is also known as First Mean Value Theorem.

$$f(x) = \tan x$$

Differentiating with respect to x :

$$f'(x) = \sec^2 x$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow \sec^2 c = \frac{\tan b - \tan a}{b - a}$$

Since c lies between a and b

$$\Rightarrow a < c < b$$

$$\Rightarrow \sec^2 a < \sec^2 c < \sec^2 b$$

$$\Rightarrow \sec^2 a < \frac{\tan b - \tan a}{b - a} < \sec^2 b$$

$$\Rightarrow \sec^2 a (b - a) < \tan b - \tan a < \sec^2 b (b - a)$$

Hence Proved

MCQ

1. Question

Mark the correct alternative in the following:

If the polynomial equation $a_0x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x + a_0 = 0$

n being a positive integer, has two different real roots α and β , then between α and β , the equation $na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \dots + a_1 = 0$ has

- A. exactly one root
- B. almost one root
- C. at least one root
- D. no root

Answer

As the polynomial, $na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \dots + a_1 = 0$ is a derivative of the polynomial $a_0x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x + a_0 = 0$ ----- (i)

Putting $x = 0$ in equation (i),

$f(0) = a_0 < 0$, {Y - Intercept of the graph is negative}

On the other hand, $\because a_n > 0$ and ' n ' is even, the leading term on x^n , is positive for only x .

For $|x|$ to be large, the term anx_n will dominate, so

$$\lim_{x \rightarrow -\infty} f(x) = +\infty$$

$$\lim_{x \rightarrow +\infty} f(x) = +\infty$$

If $\lim_{x \rightarrow +\infty} f(x) = +\infty$, there must exist

Same number $\alpha < 0$, where $f(\alpha) > 0$

$$\because f(0) = a_0 < 0,$$

$$\therefore \alpha < \beta < 0, \text{ such that } f(\beta) = 0$$

Also, there is some value $0 < \alpha$,

Where $f(a)$ & so there exists,

$$0 < b < a \text{ with } f(b) = 0$$

Additionally, the polynomial function (equation (i)) is continuous everywhere in $\mathbb{R}$ and consequently derivative in $\mathbb{R}$.

$$\therefore a_0x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x + a_0 = 0 \text{ is continuous on } \alpha, \beta \text{ and derivative on } \alpha, \beta.$$

Thus, it satisfies both the conditions of Rolle's Theorem.

As per the Rolle's Theorem, between any two roots of a function $f(x)$, there exists at least one root of its derivative.

Thus, the equation $na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \dots + a_1 = 0$ will have at least one root between α & β .

Hence, Option (C) is the answer.

2. Question

Mark the correct alternative in the following:

If $4a + 2b + c = 0$, then the equation $3ax^2 + 2bx + c = 0$ has at least one real root lying in the interval.

- A. (0, 1)
- B. (1, 2)
- C. (0, 2)

D. none of these

Answer

Let $f(x) = ax^3 + bx^2 + cx + d$ ----- (i)

$$\therefore f(0) = d$$

$$f(2) = a(2)^3 + b(2)^2 + c(2) + d$$

$$= 8a + 4b + 2c + d$$

$$= 2(4a + 2b + c) + d$$

$$\therefore 4a + 2b + c = 0 \text{ \{Given\}}$$

$$= 2(0) + d$$

$$= 0 + d$$

$$= d$$

f is continuous in closed interval $[0, 2]$ and f is derivable in the open interval $(0, 2)$.

$$\text{Also, } f(0) = f(2)$$

As per Rolle's Theorem,

$$f'(\alpha) = 0 \text{ for } 0 < \alpha < 2$$

$$f'(x) = 3ax^2 + 2bx + c$$

$$\therefore f'(\alpha) = 3a\alpha^2 + 2b(\alpha) + c$$

$$3a\alpha^2 + 2b(\alpha) + c = 0$$

Hence equation (i) has at least one root in the interval $(0, 2)$.

Thus, $f'(x)$ must have one root in the interval $(0, 2)$.

Hence, Option (C) is the answer.

3. Question

Mark the correct alternative in the following:

For the function $f(x) = x + \frac{1}{x}$, $x \in [1, 3]$, the value of c for the Lagrange's mean value theorem is

A. 1

B. $\sqrt{3}$

C. 2

D. none of these

Answer

$$f(x) = x + \frac{1}{x}$$

$$= \frac{x^2 + 1}{x}$$

It shows that $f(x)$ is continuous on $1, 3$ and derivable on $1, 3$.

So, both the conditions of Lagrange's Theorem are satisfied.

Consequently, there exists $c \in 1, 3$ such that

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{f(3) - f(1)}{2}$$

$$f(x) = \frac{x^2 + 1}{x}$$

$$f'(x) = \frac{x^2 - 1}{x^2} \left\{ \because f(x) = x + \frac{1}{x}, f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} \right\}$$

$$f(1) = \frac{x^2 + 1}{x}$$

$$= \frac{1^2 + 1}{1}$$

$$= \frac{1 + 1}{1}$$

$$= 2$$

$$f(3) = \frac{x^2 + 1}{x}$$

$$= \frac{3^2 + 1}{3}$$

$$= \frac{9 + 1}{3}$$

$$= \frac{10}{3}$$

$$f'(x) = \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{f(3) - f(1)}{2}$$

$$\therefore \frac{x^2 - 1}{x^2}$$

$$= \frac{\frac{10}{3} - 2}{2}$$

$$= \frac{\frac{10 - 6}{3}}{2}$$

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

$$\frac{x^2 - 1}{x^2} = \frac{2}{3}$$

$$3x^2 - 3 = 2x^2$$

$$x^2 = 3$$

$$x = \pm\sqrt{3} \text{ Hence, } c = \sqrt{3} \in (1, 3) \text{ such that } f'(c) = \frac{f(3)-f(1)}{3-1}.$$

Hence, Option (B) is the answer.

4. Question

Mark the correct alternative in the following:

If from Lagrange's mean value theorem, we have $\frac{f'(x) = f'(b) - f(a)}{b - a}$, then

A. $a < x_1 \leq b$

B. $a \leq x_1 < b$

C. $a < x_1 < b$

D. $a \leq x_1 \leq b$

Answer

$$\therefore f'(x) = \frac{f(b) - f(a)}{b - a}$$

In the Lagrange's Mean Value Theorem, $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

f is continuous on $[a, b]$ and differentiable on (a, b) , then there exists a real number $x \in (a, b)$

So, in the case of x_1 , $f'(x_1) = \frac{f(b)-f(a)}{b-a}$, then $x_1 \in (a, b)$

$$\therefore a < x_1 < b$$

Hence, Option (C) is the answer.

5. Question

Mark the correct alternative in the following:

Rolle's theorem is applicable in case of $\phi(x) = a^{\sin x}$, $a > 0$ in

A. any interval

B. the interval $[0, \pi]$

C. the interval $(0, \pi/2)$

D. none of these

Answer

$$\Phi(x) = a^{\sin x}, a > 0$$

Differentiating the above-mentioned function, with respect to 'x',

$$\Phi'(x) = \log a (\cos x a^{\sin x})$$

$$\therefore \Phi'(c) = \log a (\cos c a^{\sin c})$$

$$\text{Let } \Phi'(c) = 0$$

$$\log a (\cos c a^{\sin c}) = 0$$

$$\cos c a^{\sin c} = 0$$

$$\cos c = 0$$

$$\cos c = \cos \frac{\pi}{2}$$

$$\therefore c = \frac{\pi}{2}$$

Also, the above-mentioned function, is derivable and continuous on the interval $[0, \pi]$.

Thus, here Rolle's Theorem is applicable on the above mentioned function in the interval $[0, \pi]$.

Hence, Option (B) is the answer.

6. Question

Mark the correct alternative in the following:

The value of c in Rolle's theorem show $f(x) = 2x^3 - 5x^2 - 4x + 3$, is $x \in [1/3, 3]$

A. 2

B. $-\frac{1}{3}$

C. -2

D. $\frac{2}{3}$

Answer

$$f(x) = 2x^3 - 5x^2 - 4x + 3$$

$$f'(x) = 6x^2 - 10x - 4$$

$$f'(c) = 6c^2 - 10c - 4$$

$$\therefore f'(c) = 0$$

$$\therefore 6c^2 - 10c - 4 = 0$$

$$3c^2 - 5c - 2 = 0$$

$$3c^2 + c - 6c - 2 = 0$$

$$c(3c + 1) - 2(3c + 1) = 0$$

$$(3c + 1)(c - 2) = 0$$

$$c = 2 \text{ or } c = -\frac{1}{3}$$

$$\therefore c = 2 \in \left(\frac{1}{3}, 3\right)$$

Thus, as per Rolle's Theorem, $c = 2 \in \left(\frac{1}{3}, 3\right)$.

So, the required value of $c = 2$

Hence, Option (A) is the answer.

7. Question

Mark the correct alternative in the following:

The value tangent to the curve $y = x \log x$ is parallel to the chord joining the points $(1, 0)$ and (e, e) , the value of is

A. $e^{1/1-e}$

B. $e^{(e-1)(2e-1)}$

C. $\frac{2e-1}{e^{e-1}}$

D. $\frac{e-1}{e}$

Answer

$$y = x \log x$$

Differentiating the function with respect to 'x',

$$\frac{dy}{dx} = 1 + \log x$$

Slope of tangent to the curve = $1 + \log x$

And, slope of the chord joining the points, (1, 0) & (e, e)

$$m = \frac{e}{e-1}$$

The tangent to the curve is parallel to the chord joining the points, (1, 0) & (e, e)

$$\therefore m = 1 + \log x$$

$$\frac{e}{e-1} = 1 + \log x$$

$$\log x = \frac{e}{e-1} - 1$$

$$\log x = \frac{e - e + 1}{e - 1}$$

$$\log x = \frac{1}{e - 1}$$

$$x = e^{\frac{1}{e-1}}$$

Hence, Option (A) is the answer.

8. Question

Mark the correct alternative in the following:

The value of c in Rolle's theorem for the function $f(x) = \frac{x(x+1)}{e^x}$ defined on $[-1, 0]$ is

A. 0.5

B. $\frac{1+\sqrt{5}}{2}$

C. $\frac{1-\sqrt{5}}{2}$

D. -0.5

Answer

$$f(x) = \frac{\{x(x+1)\}}{e^x}$$

Differentiating the function with respect to 'x',

$$f'(x) = \frac{e^x(2x+1) - x(x+1)e^x}{(e^x)^2}$$

$$f'(x) = \frac{e^x[(2x+1) - x(x+1)]}{(e^x)^2}$$

$$f'(x) = \frac{[(2x+1) - x(x+1)]}{e^x}$$

$$f'(x) = \frac{2x+1-x^2-x}{e^x}$$

$$f'(x) = \frac{x+1-x^2}{e^x}$$

$$f'(x) = \frac{-x^2+x+1}{e^x}$$

$$\therefore f'(c) = \frac{-c^2+c+1}{e^c}$$

$$\therefore f'(c) = 0$$

$$\frac{-c^2+c+1}{e^c} = 0$$

$$-c^2+c+1 = 0$$

$$c^2-c-1 = 0$$

Using Sridharacharya Formula,

In a general equation, $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore c = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$c = \frac{1 \pm \sqrt{1+4}}{2}$$

$$c = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore c = \frac{1-\sqrt{5}}{2} \in (-1, 0)$$

So the required value of $c = \frac{1-\sqrt{5}}{2} \in (-1, 0)$.

Hence, Option (C) is the answer.

9. Question

Mark the correct alternative in the following:

The value of c in Lagrange' mean value theorem for the function $f(x) = x(x-2)$ where $x \in [1, 2]$ is

A. 1

B. $\frac{1}{2}$

C. $\frac{2}{3}$

D. $\frac{3}{2}$

Answer

$$f(x) = x(x - 2)$$

$$f(x) = x^2 - 2x$$

Since, a polynomial function is always continuous and differentiable.

As $f(x)$ is a polynomial function so it is always continuous on 1, 2 and differentiable on 1, 2.

$\therefore f(x)$ satisfies both the conditions of Lagrange's Theorem on 1, 2.

So, a real number has to exist $c \in 1, 2$, such that

$$f'(c) = \frac{f(2) - f(1)}{(2 - 1)}$$

$$= \frac{f(2) - f(1)}{1}$$

$$\because f(x) = x^2 - 2x$$

$$f'(x) = 2x - 2$$

$$\therefore f(1) = 1^2 - 2(1)$$

$$= 1 - 2$$

$$= -1$$

$$f(2) = 2^2 - 2(2)$$

$$= 4 - 4$$

$$= 0$$

$$\therefore f'(x) = \frac{f(2) - f(1)}{1}$$

$$f'(x) = \frac{0 - (-1)}{1}$$

$$= \frac{0 + 1}{1}$$

$$= 1$$

$$\text{So, } 2x - 2 = 1$$

$$2x = 3$$

$$x = \frac{3}{2}$$

$$\therefore c = \frac{3}{2} \in (1, 2)$$

Hence, Option (D) is the answer.

10. Question

Mark the correct alternative in the following:

The value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in the interval $[0, \sqrt{3}]$ is

A. 1

B. -1

C. $\frac{3}{2}$

D. $\frac{1}{3}$

Answer

$$f(x) = x^3 - 3x$$

The above mentioned polynomial function is continuous and derivable in $\mathbb{R}$.

$\therefore$ the function is continuous on $[0, \sqrt{3}]$ and derivable on $[0, \sqrt{3}]$.

Differentiating the function with respect to x ,

$$f(x) = x^3 - 3x$$

$$f'(x) = 3x^2 - 3$$

$$\therefore f'(c) = 3c^2 - 3$$

$$\therefore f'(c) = 0$$

$$3c^2 - 3 = 0$$

$$c^2 - 1 = 0$$

$$c^2 = 1$$

$$c = \pm 1$$

Hence, $c = 1 \in [0, \sqrt{3}]$, as per the condition of Rolle's Theorem.

The required value is $c = 1$.

Hence, Option (A) is the answer.

11. Question

Mark the correct alternative in the following:

If $f(x) = e^x \sin x$ in $[0, \pi]$, then c in Rolle's theorem is

A. $\frac{\pi}{6}$

B. $\frac{\pi}{4}$

C. $\frac{\pi}{2}$

D. $\frac{3\pi}{4}$

Answer

$$\text{As, } f(x) = e^x \sin x$$

Differentiating the function with respect to 'x',

$$f'(x) = e^x \cos x + \sin x e^x$$

$$\therefore f'(c) = e^c \cos c + \sin c e^c$$

As, $e^x \cos x$ is continuous and derivable in $\mathbb{R}$.

$\therefore$ it is continuous on $[0, \pi]$ and derivable on $(0, \pi)$.

$$f(0) = e^0 \sin(0)$$

$$= 0$$

$$f(\pi) = e^\pi \sin \pi$$

$$= e^\pi(0)$$

$$= 0$$

$$\therefore f'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(\pi) - f(0)}{\pi - 0}$$

$$= \frac{0 - 0}{\pi}$$

$$= 0$$

$$e^c \cos c + \sin c e^c = 0$$

$$e^c (\cos c + \sin c) = 0$$

$$\cos c + \sin c = 0 \text{ ----- (i)}$$

$$\cos c = -\sin c$$

$$\cos c = -\cos\left(\frac{\pi}{2} - c\right)$$

$$\cos c = \cos\left(\pi + \left(\frac{\pi}{2} - c\right)\right)$$

$$\cos c = \cos\left(\pi + \frac{\pi}{2} - c\right)$$

$$\cos c = \cos\left(\frac{3\pi}{2} - c\right)$$

$$c = \frac{3\pi}{2} - c$$

$$2c = \frac{3\pi}{2}$$

$$c = \frac{3\pi}{4}$$

$$\therefore c = \frac{3\pi}{4} \in (0, \pi)$$

Hence, Option (D) is the answer.

Very short answer

1. Question

If $f(x) = Ax^2 + Bx + C$ is such that $f(a) = f(b)$, then write the value of c in Rolle's theorem.

Answer

$$f(x) = Ax^2 + Bx + C$$

Differentiating the above-mentioned function with respect to 'x',

$$f'(x) = 2Ax + B$$

$$f'(c) = 2Ac + B$$

$$f'(c) = 0$$

$$\therefore 2Ac + B = 0$$

$$c = -\frac{B}{2A} \text{ ----- (i)}$$

$$\text{As } f(a) = f(b),$$

$$f(a) = Aa^2 + Ba + C$$

$$f(b) = Ab^2 + Bb + C$$

$$Aa^2 + Ba + C = Ab^2 + Bb + C$$

$$Aa^2 + Ba = Ab^2 + Bb$$

$$A(a^2 - b^2) + B(a - b) = 0$$

$$A(a + b)(a - b) + B(a - b) = 0$$

$$(a - b)\{A(a + b) + B\} = 0$$

$$a = b, A = -\frac{B}{a+b}$$

$$a + b = -\frac{B}{A} \text{ \{As } a \neq b\}}$$

From equation (i)

$$c = \frac{a + b}{2}$$

Hence the required value is $= \frac{a+b}{2}$.

2. Question

State Rolle's theorem.

Answer

Rolle's Theorem is stated as below: -

Let 'f', be a real valued function defined on the closed interval a, b such that

i. It is continuous in the closed interval $[a, b]$.

ii. It is differentiable in the open interval (a, b) .

iii. $f(a) = f(b)$

Then there exists a real number $c \in (a, b)$ such that $f'(c) = 0$.

3. Question

State Lagrange's mean value theorem.

Answer

Lagrange's Mean Value Theorem is stated as below :-

Let $f(x)$ be a function defined on a, b such that

i. It is continuous on a, b and

ii. It is differentiable on a, b .

Then there exists a real number $c \in a, b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

4. Question

If the value of c prescribed in Rolle's theorem for the function $f(x) = 2x(x-3)^n$ on the interval $\left[0, 2\sqrt{3}\right]$ is $\frac{3}{4}$, write the value of n (a positive integer).

Answer

$$f(x) = 2x(x-3)^n$$

Differentiating the above-mentioned function with respect to 'x',

$$f'(x) = 2 [xn(x-3)^{n-1} + (x-3)^n]$$

$$f'(x) = 2(x-3)^n \left[\frac{xn}{x-3} + 1 \right]$$

$$\therefore f'(c) = 2(c-3)^n \left[\frac{cn}{c-3} + 1 \right]$$

$$\because f'\left(\frac{3}{4}\right) = 0$$

$$\therefore 2 - \left(\frac{9}{4}\right)^n \left[\frac{\frac{3}{4}n}{-\frac{3}{4}} + 1 \right] = 0$$

$$2 - \left(\frac{9}{4}\right)^n \left[-\frac{n}{3} + 1 \right] = 0$$

$$-\frac{n}{3} + 1 = 0$$

$$\frac{(-n+3)}{3} = 0$$

$$-n + 3 = 0$$

$$-n = -3$$

$$n = 3$$

Hence, the required value of 'n' is 3.

5. Question

Find the value of c prescribed by Lagrange's mean value theorem for the function $f(x) = \sqrt{x^2 - 4}$ defined on $[2, 3]$.

Answer

$$f(x) = \sqrt{x^2 - 4}$$

$f(x)$ will exist, if

$$x^2 - 4 \geq 0$$

$$x^2 \geq 4$$

$$\therefore x \leq -2 \text{ or } x \geq 2$$

$\therefore$ for each $x \in [2, 3]$, the function $f(x)$ has a unique definite value, $f(x)$ is continuous on $(2, 3)$.

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x^2-4}} (2x) \\ &= \frac{x}{2\sqrt{x^2-4}} \end{aligned}$$

Exists for all $x \in (2, 3)$.

So, $f(x)$ is differentiable on $(2,3)$.

Hence, both the conditions of Lagrange's Theorem are satisfied.

Consequently, there exists $c \in (2, 3)$ such that,

$$\begin{aligned} f'(c) &= \frac{f(3) - f(2)}{3 - 2} \\ &= \frac{f(3) - f(2)}{1} \end{aligned}$$

$$f(x) = \sqrt{x^2 - 4}$$

$$f(3) = \sqrt{3^2 - 4}$$

$$= \sqrt{9 - 4}$$

$$= \sqrt{5}$$

$$f(2) = \sqrt{2^2 - 4}$$

$$= \sqrt{4 - 4}$$

$$= 0$$

$$f'(x) = \frac{x}{\sqrt{x^2-4}}$$

$$\therefore f'(x) = \frac{f(3) - f(2)}{3 - 2}$$

$$f'(x) = \frac{\sqrt{5} - 0}{1}$$

$$\frac{x}{\sqrt{x^2-4}} = \sqrt{5}$$

Squaring both sides,

$$\frac{x^2}{(\sqrt{x^2-4})^2} = (\sqrt{5})^2$$

$$\frac{x^2}{x^2-4} = 5$$

$$5x^2 - 20 - x^2 = 0$$

$$4x^2 = 20$$

$$x^2 = 5$$

$$x = \pm \sqrt{5}$$

$$\text{Hence, } c = \sqrt{5} \in (2, 3) \text{ such that } f'(c) = \frac{f(3)-f(2)}{3-2}$$

Hence the above explanation verifies the Lagrange's Theorem.

1. Question

What are the values of 'a' for which $f(x) = a^x$ is increasing on $\mathbb{R}$?

Answer

$$f(x) = a^x$$

$$f'(x) = a^x \log a$$

∴ $f(x)$ is increasing on $\mathbb{R}$.

$$\therefore f'(x) > 0$$

$$\therefore a^x \log a > 0$$

∴ Logarithmic function is defined for positive values of a.

$$\therefore a > 0$$

$$\therefore a^x > 0$$

$$\therefore a^x \log a > 0$$

∴ It can be possible when $\log a^x > 0$ & $\log a > 0$ or $a^x < 0$ & $\log a < 0$.

$$\log a > 0$$

$$\therefore a > 1$$

Hence, $f(x)$ is increasing when $a > 1$.

2. Question

What are the values of 'a' for which $f(x) = a^x$ is decreasing on $\mathbb{R}$?

Answer

$$f(x) = a^x$$

$$f'(x) = a^x \log a$$

∴ $f(x)$ is decreasing on $\mathbb{R}$.

$$\therefore f'(x) < 0, \forall x \in \mathbb{R}$$

$$\therefore a^x \log a < 0, \forall x \in \mathbb{R}$$

∴ Logarithmic function is not defined for negative values of a.

$$\therefore a^x > 0$$

∴ $a^x \log a < 0$ can be possible when $\log a < 0, \forall x \in \mathbb{R}$.

$$\therefore 1 > a > 0$$

Hence the function $f(x)$, whose values are $1 > a > 0$.

3. Question

Write the set of values of 'a' for which $f(x) = \log_a x$ is increasing in its domain.

Answer

$$f(x) = \log_a x$$

Let $x_1, x_2 \in (0, \infty)$ such that $x_1 < x_2$.

∴ the function here is a logarithmic function, so either $a > 1$ or $1 > a > 0$.

Case - 1

Let $a > 1$

$$x_1 < x_2$$

$$\therefore \log_a x_1 < \log_a x_2$$

$$\therefore f(x_1) < f(x_2)$$

$$\therefore x_1 < x_2 \text{ \& } f(x_1) < f(x_2), \forall x_1, x_2 \in (0, \infty)$$

Hence, $f(x)$ is increasing on $(0, \infty)$.

Case - 2

Let, $1 > a > 0$

$$x_1 < x_2$$

$$\therefore \log_a x_1 > \log_a x_2$$

$$\therefore f(x_1) > f(x_2)$$

$$\therefore x_1 < x_2 \text{ \& } f(x_1) > f(x_2), \forall x_1, x_2 \in (0, \infty)$$

Thus, for $a > 1$, $f(x)$ is increasing in its domain.

4. Question

Write the set of values of 'a' for which $f(x) = \log_a x$ is decreasing in its domain.

Answer

$$f(x) = \log_a x$$

Domain of the above mentioned function is $(0, \infty)$

Let $x_1, x_2 \in (0, \infty)$ such that $x_1 < x_2$.

∴ the function here is a logarithmic function, so either $a > 1$ or $1 > a > 0$.

Case - 1

Let $a > 1$

$$x_1 < x_2$$

$$\therefore \log_a x_1 < \log_a x_2$$

$$\therefore f(x_1) < f(x_2)$$

$$\therefore x_1 < x_2 \text{ \& } f(x_1) < f(x_2), \forall x_1, x_2 \in (0, \infty)$$

Hence, $f(x)$ is increasing on $(0, \infty)$.

Case - 2

Let $1 > a > 0$

$$x_1 < x_2$$

$$\therefore \log_a x_1 > \log_a x_2$$

$$\therefore f(x_1) > f(x_2)$$

$$\therefore x_1 < x_2 \text{ \& } f(x_1) > f(x_2), \forall x_1, x_2 \in (0, \infty)$$

Hence, $f(x)$ is decreasing on $(0, \infty)$.

Thus, for $1 > a > 0$, $f(x)$ is decreasing in its domain.

5. Question

Find 'a' for which $f(x) = a(x + \sin x) + a$ is increasing on $\mathbb{R}$.

Answer

$$f(x) = a(x + \sin x) + a$$

$$f'(x) = a(1 + \cos x) + 0$$

$$f'(x) = a(1 + \cos x)$$

For $f(x)$, to be increasing, it must have,

$$f'(x) > 0$$

$$\therefore a(1 + \cos x) > 0 \text{ ----- (i)}$$

$$\therefore -1 \leq \cos x \leq 1, \forall x \in \mathbb{R}$$

$$\therefore 0 \leq (1 + \cos x) \leq 2, \forall x \in \mathbb{R}$$

$$\therefore a > 0 \text{ \{From eq. (i)\}}$$

$$\therefore a \in (0, \infty)$$

Hence the required set of values is $a \in (0, \infty)$.

6. Question

Find the values of 'a' for which the function $f(x) = \sin x - ax + 4$ is increasing function on $\mathbb{R}$.

Answer

$$f(x) = \sin x - ax + 4$$

$$f'(x) = \cos x - a + 0$$

$$f'(x) = \cos x - a$$

$$\therefore f(x) \text{ is increasing on } \mathbb{R}.$$

$$f'(x) > 0$$

$$\therefore \cos x - a > 0$$

$$\cos x > a$$

$$\therefore \cos x \geq -1, \forall x \in \mathbb{R}$$

$$\therefore a < -1$$

$$\therefore a \in (-\infty, -1)$$

Hence the required set of values is $a \in (-\infty, -1)$.

7. Question

Find the set of values of 'b' for which $f(x) = b(x + \cos x) + 4$ is decreasing on $\mathbb{R}$.

Answer

$$f(x) = b(x + \cos x) + 4$$

$$f'(x) = b(1 - \sin x) + 0$$

$$f'(x) = b(1 - \sin x)$$

$\therefore f(x)$ is decreasing on $\mathbb{R}$.

$$\therefore f'(x) < 0$$

$$b(1 - \sin x) < 0$$

$$\therefore \sin x \leq 1$$

$$1 - \sin x \geq 0$$

$$\therefore b(1 - \sin x) < 0 \text{ \& } 1 - \sin x \geq 0$$

$$\therefore b < 0$$

$$\therefore b \in (-\infty, 0)$$

Hence the required set of values is $b \in (-\infty, 0)$.

8. Question

Find the set of values of a for which $f(x) = x + \cos x + ax + b$ is increasing on $\mathbb{R}$.

Answer

$$f(x) = x + \cos x + ax + b$$

$$f'(x) = 1 - \sin x + a + 0$$

$$f'(x) = 1 - \sin x + a$$

For, $f(x)$ to be increasing, it must have

$$f'(x) > 0$$

$$\therefore 1 - \sin x + a > 0$$

$$1 > \sin x - a$$

$$\sin x < a + 1$$

$\therefore$ the maximum value of $\sin x$ is 1.

$$\text{Also, } 1 < a + 1$$

$$a > 0$$

$$\therefore a \in (0, \infty)$$

Hence the required set of values is $a \in (0, \infty)$.

9. Question

Write the set of values of k for which $f(x) = kx - \sin x$ is increasing on $\mathbb{R}$.

Answer

$$f(x) = kx - \sin x$$

$$f'(x) = k - \cos x$$

For, $f(x)$ to be increasing, it must have

$$f'(x) > 0$$

$$\therefore k - \cos x > 0$$

$$k > \cos x$$

$$\cos x < k$$

$\therefore$ the minimum value of $\cos x$ is 1.

Also, $\cos x < k$

$\therefore$ The minimum value of k is 1.

$$\therefore k \in (1, \infty)$$

Hence the required set of values is $k \in (1, \infty)$.

10. Question

If $g(x)$ is a decreasing function on $\mathbb{R}$ and $f(x) = \tan^{-1} \{g(x)\}$. State whether $f(x)$ is increasing or decreasing on $\mathbb{R}$.

Answer

$\therefore g(x)$ is decreasing on $\mathbb{R}$.

$f(x) = \tan^{-1}x$ is an increasing function.

$$f \circ g(x) = f(g(x)) = \tan^{-1}(g(x))$$

is a decreasing function

Composite of two functions,

$f(x)$	$g(x)$	Composite
$\uparrow$	$\uparrow$	$\uparrow$
$\downarrow$	$\downarrow$	$\uparrow$
$\uparrow$	$\downarrow$	$\downarrow$
$\downarrow$	$\uparrow$	$\downarrow$

Glossary –

$\uparrow$ - Increasing

$\downarrow$ - For decreasing $f(x)$.

$$\therefore x_1 < x_2$$

$$\therefore g(x_1) > g(x_2)$$

Applying, $\tan^{-1}$ on both the sides, of the mentioned equation,

$$\therefore \tan^{-1} \{g(x_1)\} > \tan^{-1} \{g(x_2)\}$$

$$\therefore f(x_1) > f(x_2)$$

Hence it is decreasing on R.

11. Question

Write the set of values of a for which the function $f(x) = ax + b$ is decreasing for all $x \in \mathbb{R}$.

Answer

$$f(x) = ax + b$$

$$f'(x) = a + 0$$

$$f'(x) = a$$

For, $f(x)$ to be decreasing, it must have

$$f'(x) < 0$$

$$\therefore a < 0$$

$$\therefore a \in (-\infty, 0)$$

Hence the required set of values is $a \in (-\infty, 0)$.

12. Question

Write the interval in which $f(x) = \sin x + \cos x$, $x \in [0, \pi/2]$ is increasing.

Answer

$$f(x) = \sin x + \cos x, x \in [0, \frac{\pi}{2}]$$

$$f'(x) = \cos x - \sin x$$

For, $f(x)$ to be increasing, it must have

$$f'(x) > 0$$

$$\cos x - \sin x > 0$$

$$\sin x < \cos x$$

$$\frac{\sin x}{\cos x} < 1$$

$$\tan x < 1$$

$$\tan x < \tan \frac{\pi}{4}$$

$$\therefore x \in \left[0, \frac{\pi}{4}\right)$$

Hence, the required interval is $x \in \left[0, \frac{\pi}{4}\right)$.

13. Question

State whether $f(x) = \tan x - x$ is increasing or decreasing its domain.

Answer

$$f(x) = \tan x - x$$

$$f'(x) = \sec^2 x - 1$$

$$\sec^2 x - 1 \geq 0$$

$$\sec^2 x \geq 1$$

$$\therefore 1 + \tan^2 x = \sec^2 x$$

$$\therefore 1 + \tan^2 x \geq 1$$

$$\tan^2 x \geq 0$$

$$\therefore \tan^2 x \geq 0 \quad \forall x \in [0, 2\pi]$$

$\therefore f(x)$ is increasing its domain.

14. Question

Write the set of values of a for which $f(x) = \cos x + a^2 x + b$ is strictly increasing on $\mathbb{R}$.

Answer

$$f(x) = \cos x + a^2 x + b$$

$$f'(x) = -\sin x + a^2 + 0$$

$$f'(x) = a^2 - \sin x$$

$\therefore f(x)$ is strictly increasing on $\mathbb{R}$

$$\therefore f'(x) > 0, \quad \forall x \in \mathbb{R}$$

$$\therefore a^2 - \sin x > 0, \quad \forall x \in \mathbb{R}$$

$$\therefore a^2 > \sin x, \quad \forall x \in \mathbb{R}$$

$\therefore$ Maximum value of $\sin x$ is 1.

$$\therefore a^2 > \sin x, \quad a^2 \text{ is always greater than } 1.$$

$$a^2 > 1$$

$$a^2 - 1 > 0$$

$$(a + 1)(a - 1) > 0$$

$$\therefore a \in (-\infty, -1) \cup (1, \infty)$$